A.1.1)

$$m\,\mathcal{L}\left[\frac{d^2x(t)}{dt^2}\right] + K_d\,\mathcal{L}\left[\frac{dx(t)}{dt}\right] - K_s\,\mathcal{L}[x(t)] = 0$$

$$\Rightarrow m\left(s^2X(s) - s\right) + K_d\left(sX(s) - 1\right) - K_s X(s) = 0$$

$$\Rightarrow m s^2 X(s) + K_d s X(s) - K_s X(s) - ms - K_d = 0 \quad (1)'$$

A.1.2) $X(s)\left(ms^2 + K_d s - K_s\right) = sm + K_d$

$$\Rightarrow X(s) = \frac{sm + K_d}{ms^2 + K_d s - K_s} = \frac{N(s)}{D(s)}$$

$N(s)$ has order 1

$D(s)$ has order 2

A.1.3)

$$s_{1,2} = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_s}}{2m} = -\frac{K_d}{2m} \pm \sqrt{\frac{K_d^2}{4m^2} - \frac{K_s}{m}}$$

$$X(s) = \frac{sm + K_d}{m(s - s_1)(s - s_2)}$$

A.1.4)

$$X(s) = \frac{ms}{m\left(s - j\sqrt{\frac{K_s}{m}}\right)\left(s + j\sqrt{\frac{K_s}{m}}\right)}$$

$s_1 = j\sqrt{\frac{k_s}{m}}$ $\quad$ $s_2 = -j\sqrt{\frac{k_s}{m}}$

they are imaginary numbers and $s_1 = s_2^*$
they are conjugate

A.1.5)

$$X(s) = \frac{s}{s^2 + \frac{k_s}{m}}$$

$$x(t) = \cos\left(\sqrt{\frac{k_s}{m}}\, t\right)$$

If the damping force is zero then the object keeps oscillating between +1 an -1 because nothing can dissipate energy.

A.1.6) since $s_{1,2} = \frac{-k_d \pm \sqrt{k_d^2 - 4mk_s}}{2m}$

if $k_d^2 - 4mk_s > 0$ then the two roots are real.

A.1.7)

$$X(s) = \frac{s}{(s-s_1)(s-s_2)} = \frac{A}{s-s_1} + \frac{B}{s-s_2}$$

$$A = \lim_{s \to s_1} X(s)(s-s_1) = \frac{s_1}{(s_1 - s_2)}$$

$$B = \lim_{s \to s_2} X(s)(s-s_2) = \frac{s_2}{s_2 - s_1}$$

$$s_1 - s_2 = 2\sqrt{\frac{K_d^2}{4m^2} - \frac{K_s}{m}} = \sqrt{\frac{K_d^2}{m^2} - \frac{4K_s}{m}} = -(s_2 - s_1)$$

$$X(s) = \frac{s_1}{s_1 - s_2}\frac{1}{(s - s_1)} + \frac{s_2}{s_2 - s_1}\frac{1}{s - s_2}$$

$$x(t) = \frac{s_1}{s_1 - s_2} e^{s_1 t} + \frac{s_2}{s_2 - s_1} e^{s_2 t}$$

note that s_1 and s_2 are both negative so the object will stop.

It will stop when $s_1 e^{s_1 t} = s_2 e^{s_2 t}$

$$\frac{s_1}{s_2} = e^{(s_2 - s_1)t} \Rightarrow \ln\frac{s_1}{s_2} = (s_2 - s_1)t$$

$$\Rightarrow \boxed{t^* = \frac{\ln s_1 - \ln s_2}{s_2 - s_1}}$$

A.1.8) $s_1 = \underbrace{-\frac{K_d}{2m}}_{=\alpha} + j\underbrace{\sqrt{\frac{K_s}{m} - \frac{K_d^2}{4m^2}}}_{=\omega_0}$ $\quad s_2 = s_1^* = \alpha - j\omega_0$

$$X(s) = \frac{s}{(s - s_1)(s - s_1^*)} = \frac{A}{s - s_1} + \frac{B}{s - s_1^*}$$

$$A = \lim_{s \to s_1} X(s)(s - s_1) = \frac{s_1}{s_1 - s_1^*} = \frac{s_1}{2j\omega_0}$$

$$B = \lim_{s \to s_1^*} X(s)(s - s_1^*) = s_1^*/(-2j\omega_0)$$

$$X(s) = \frac{s_1}{2j\omega_0}\,\frac{1}{s-s_1} - \frac{s_1^*}{2j\omega_0}$$

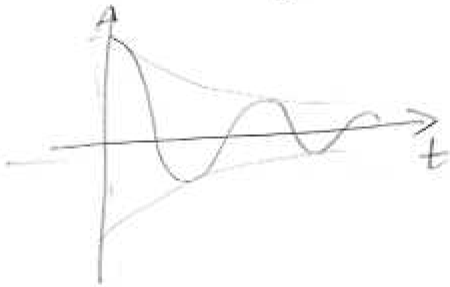
$$x(t) = \frac{s_1}{2j\omega_0}\,e^{s_1 t} - \frac{s_1^*}{2j\omega_0}\,e^{s_1^* t} =$$

$$= \left|\frac{s_1}{2j\omega_0}\right| e^{s_1 t + j\measuredangle(s_1/2j\omega_0)} - \left|\frac{s_1^*}{2j\omega_0}\right| e^{s_1^* t - j\measuredangle(s_1^*/2j\omega_0)}$$

$$= \left|\frac{s_1}{2j\omega_0}\right| 2\,\mathrm{Im}\left\{ e^{s_1 t + j\measuredangle(s_1/2j\omega_0)} \right\} =$$

$$= \underbrace{\left|\frac{s_1}{2j\omega_0}\right|}_{A=} 2\,\mathrm{Im}\left\{ e^{\alpha t}\, e^{j(\omega_0 t + \overbrace{\measuredangle(s_1/2j\omega_0)}^{=\varphi})} \right\} =$$

$$= 2A\,e^{\alpha t}\sin(\omega_0 t + \varphi) \leftarrow$$



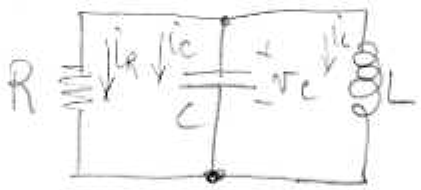
$$\left|\frac{s_1}{2j\omega_0}\right| = \left|\frac{s_1(-j\omega_0)}{2\omega_0^2}\right| = \frac{1}{2\omega_0^2}\left|(\alpha+j\omega_0)j\omega_0\right| =$$

$$= \frac{1}{2\omega_0^2}\left|\alpha j\omega_0 - \omega_0^2\right| = \frac{1}{2\omega_0^2}\sqrt{\alpha^2\omega_0^2 + \omega_0^4} =$$

$$= \frac{1}{2}\sqrt{\alpha^2+\omega_0^2} = \frac{1}{2}\sqrt{\frac{K_d^2}{4m^2} + \frac{K_s}{m} - \frac{K_d^2}{4m^2}} = \frac{1}{2}\sqrt{\frac{K_s}{m}}$$

$$\measuredangle\frac{s_1}{2j\omega_0} = \measuredangle s_1 - \pi/2 = \mathrm{tg}^{-1}\frac{\sqrt{K_s/m - K_d^2/4m^2}}{K_d/2m} - \pi/2$$

A.2)



$v_c(0) = 1$ $\quad$ $i_c(0) = 0$

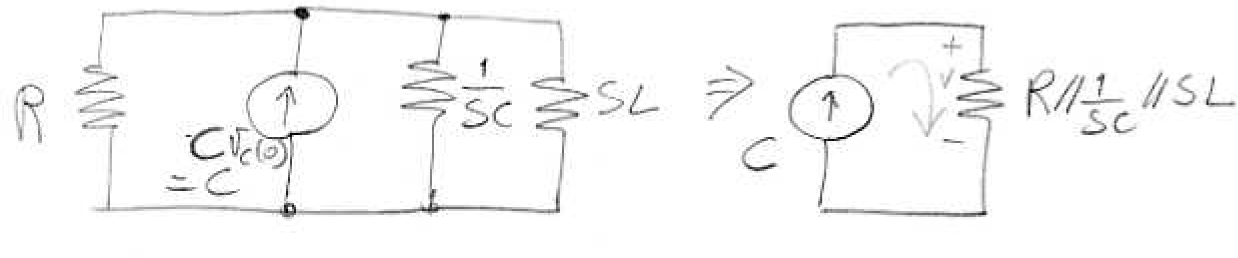
$$i_R + i_c + i_L = 0 \Rightarrow \frac{v_R}{R} + C\frac{dv_c}{dt} + \frac{1}{L}\int v_L \, dt = 0$$

$$v_R = v_c = v_L = v$$

$$\frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2} + \frac{1}{L}v = 0 \qquad v(0) = 1, \ \frac{dv}{dt}\Big|_{t=0} = 0$$

R is like the dumping force. C and L are the spring. Stretching the spring means charging the capacitor while compressing the spring means charging the inductor. You can now go ahead with the same computation.

A.2.2)



$$V(s) = C \quad R /\!/ \frac{1}{sC} /\!/ sL =$$

$$= C\left(\frac{R}{sCR+1}\right) /\!/ sL = \frac{\frac{CRsL}{sCR+1}}{\frac{R}{sCR+1} + sL} =$$

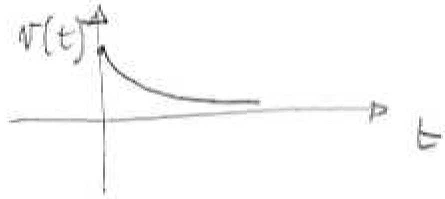
$$= C \frac{RsL}{R + (sL)(sCR+1)} = CRL \frac{s}{s^2LCR + sL + R}$$

$$= \frac{s}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

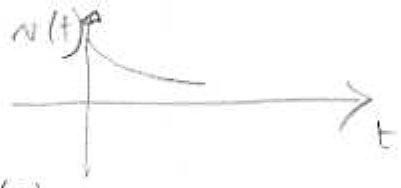
$$s_{1,2} = \frac{-1/\tau \pm \sqrt{\frac{1}{\tau^2} - \frac{4}{LC}}}{2} = \qquad \tau = RC$$

$$= -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - \frac{1}{LC}}$$

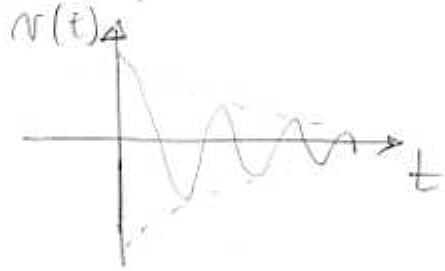
1) $\frac{1}{4\tau^2} - \frac{1}{LC} = 0 \rightarrow$



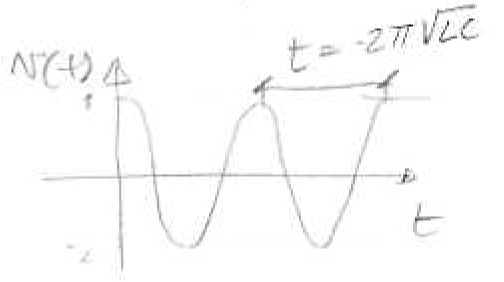
2) $\frac{1}{4\tau^2} - \frac{1}{LC} > 0 \rightarrow$



3) $\frac{1}{4\tau^2} - \frac{1}{LC} < 0 \rightarrow$

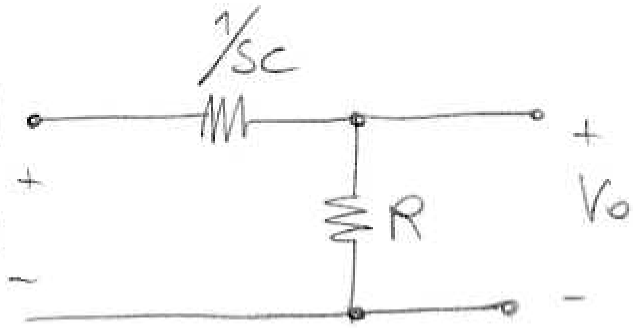


Notice that if $R \to \infty$ (R is not present which is like the damping force is not present) then $\tau \to 0$ and $s_{12} = \pm j \frac{1}{\sqrt{LC}}$



B)

B.1)



$$V_o = V_i \frac{R}{R + \frac{1}{SC}} = V_i \frac{SCR}{SCR + 1}$$

$$\Rightarrow F(S) = \frac{SCR}{SCR+1}$$

B.2)

$$F(j\omega_o) = \frac{j\omega_o CR}{j\omega_o CR + 1} \qquad CR = \tau$$

$$= \frac{j\omega_o \tau}{j\omega_o \tau + 1} = \frac{j\omega_o \tau (1 - j\omega_o \tau)}{(\omega_o \tau)^2 + 1} = \frac{(\omega_o \tau)^2 + j\omega_o \tau}{(\omega_o \tau)^2 + 1}$$

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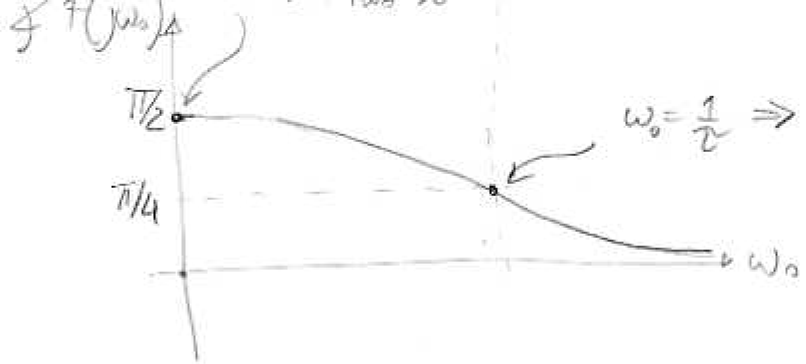
$$|F(j\omega_0)| = \frac{\sqrt{(\omega_0\tau)^4 + (\omega_0\tau)^2}}{(\omega_0\tau)^2+1} = \frac{(\omega_0\tau)\sqrt{(\omega_0\tau)^2+1}}{(\omega_0\tau)^2+1} =$$

$$= \frac{\omega_0\tau}{\sqrt{(\omega_0\tau)^2+1}}$$

$$\measuredangle F(j\omega_0) = tg^{-1}\frac{\omega_0\tau}{(\omega_0\tau)^2} = tg^{-1}\frac{1}{\omega_0\tau}$$

$$\left.\frac{1}{\omega_0\tau}\right|_{\omega_0\to 0} = \infty$$



$$\omega_0 = \frac{1}{\tau} \Rightarrow \frac{1}{\omega_0\tau} = 1 \Rightarrow tg^{-1} 1 = \pi/4$$