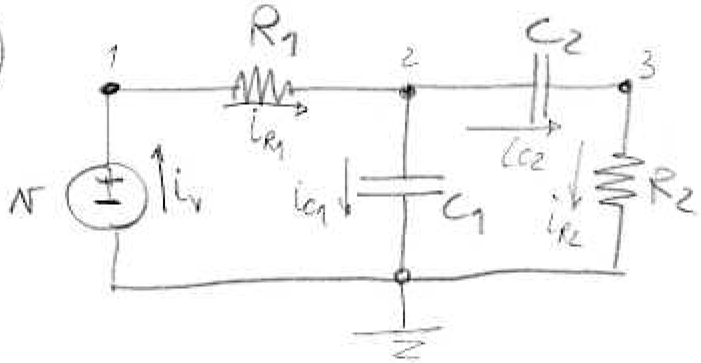
A.1)



$$\begin{cases} (1) & i_v - i_{R_1} = 0 \\ (2) & i_{R_1} - i_{C_1} - i_{C_2} = 0 \\ (3) & i_{C_2} - i_{R_2} = 0 \end{cases} \qquad \begin{cases} i_v - \dfrac{v_1 - v_2}{R_1} = 0 \\ \dfrac{v_1 - v_2}{R_1} - C_1 \dfrac{dv_2}{dt} - C_2 \dfrac{d(v_2 - v_3)}{dt} = 0 \\ C_2 \dfrac{d(v_2 - v_3)}{dt} - \dfrac{v_3}{R_2} = 0 \end{cases}$$

$v_1 = v$

$$\begin{cases} (1) & i_v - \dfrac{v - v_2}{R_1} = 0 \\ (2) & \dfrac{v - v_2}{R_1} - C_1 \dfrac{dv_2}{dt} - C_2 \dfrac{d(v_2 - v_3)}{dt} = 0 \\ (3) & C_2 \dfrac{d(v_2 - v_3)}{dt} - \dfrac{v_3}{R_2} = 0 \end{cases}$$

A.2) taking derivative of (2):

$$\frac{1}{R_1}\frac{dv}{dt} - \frac{1}{R_1}\frac{dv_2}{dt} - C_1 \frac{d^2 v_2}{dt^2} - C_2 \frac{d^2 v_2}{dt^2} + C_2 \frac{d^2}{dt^2} v_3 = 0$$

$$\Rightarrow \frac{1}{R_1}\frac{dv}{dt} - \frac{1}{R_1}\frac{dv_2}{dt} - (C_1 + C_2)\frac{d^2 v_2}{dt^2} + C_2 \frac{d^2}{dt^2} v_3 = 0$$

from ③:

$$C_2\frac{dv_2}{dt}-C_2\frac{dv_3}{dt}-\frac{v_3}{R_2}=0 \Rightarrow \frac{dv_2}{dt}=\frac{dv_3}{dt}+\frac{1}{R_2C_2}v_3$$

and also taking derivative

$$\frac{d^2v_2}{dt^2}=\frac{d^2v_3}{dt^2}+\frac{1}{R_2C_2}\frac{dv_3}{dt}$$

Now we substitute $\frac{dv_2}{dt}$ and $\frac{d^2v_2}{dt^2}$ in ② to obtain an equation in v_3 only:

$$\frac{1}{R_1}\frac{dv}{dt}-\frac{1}{R_1}\frac{dv_3}{dt}-\frac{1}{R_1R_2C_2}v_3-(C_1+C_2)\frac{d^2v_3}{dt}-\frac{C_1+C_2}{R_2C_2}\frac{dv_3}{dt}+$$

$$+C_2\frac{d^2}{dt^2}v_3=0 \quad \Rightarrow$$

$$\Rightarrow -C_1\frac{d^2v_3}{dt^2}-\left(\frac{1}{R_1}+\frac{C_1+C_2}{R_2C_2}\right)\frac{dv_3}{dt}-\frac{v_3}{R_1R_2C_2}+\frac{1}{R_1}\frac{dv}{dt}=0 \qquad ②'$$

A.3) $v_3=A\sin(w_0t)+B\cos(w_0t)$, $v(t)=\sin(w_0t)$

$$-C_1\left(-Aw_0^2\sin(w_0t)-Bw_0^2\cos(w_0t)\right)-\left(\frac{1}{R_1}+\frac{C_1+C_2}{R_2C_2}\right)\left(Aw_0\cos(w_0t)-Bw_0\sin(w_0t)\right)$$

$$-\frac{1}{R_1R_2C_2}\left[A\sin(w_0t)+B\cos(w_0t)\right]+\frac{1}{R_1}w_0\cos(w_0t)=0$$

$$\sin(w_0t)\left[C_1Aw_0^2+Bw_0\left(\frac{1}{R_1}+\frac{C_1+C_2}{R_2C_2}\right)-\frac{A}{R_1R_2C_2}\right]+$$

$$\cos(\omega_0 t)\left[C_1 B\omega_0^2 - \left(\frac{1}{R_1} + \frac{C_1+C_2}{R_2C_2}\right)\omega_0 A - \frac{B}{R_1R_2C_2} + \frac{\omega_0}{R_1}\right] = 0$$ 3 sol 2

$$\begin{cases} C_1 A\omega_0^2 + B\omega_0\left(\frac{1}{R_1} + \frac{C_1+C_2}{R_2C_2}\right) - \frac{A}{R_1R_2C_2} = 0 \\ C_1 B\omega_0^2 - \left(\frac{1}{R_1} + \frac{C_1+C_2}{R_2C_2}\right)\omega_0 A - \frac{B}{R_1R_2C_2} + \frac{\omega_0}{R_1} = 0 \end{cases}$$

$$\begin{cases} R_1R_2C_1C_2A\omega_0^2 + B\omega_0(R_2C_2 + R_1C_1 + R_1C_2) - A = 0 \\ R_1R_2C_1C_2B\omega_0^2 - (R_2C_2 + R_1C_1 + R_1C_2)\omega_0 A - B + \omega_0 R_2C_2 = 0 \end{cases}$$

$$\tau_1 = R_1C_1 \;;\; \tau_2 = R_2C_2 \;;\; (\tau_1 + \tau_2 + R_1C_2) = \tau^*$$

$$\begin{cases} \tau_1\tau_2\omega_0^2 A + B\omega_0\tau^* - A = 0 \longrightarrow B = A\frac{(1-\tau_1\tau_2\omega_0^2)}{\omega_0\tau^*} \\ \tau_1\tau_2\omega_0^2 B - \tau^*\omega_0 A - B = -\omega_0\tau_2 \end{cases}$$

$$\downarrow$$

$$\tau_1\tau_2\omega_0^2\frac{A(1-\tau_1\tau_2\omega_0^2)}{\omega_0\tau^*} - \tau^*\omega_0 A - A\frac{(1-\tau_1\tau_2\omega_0^2)}{\omega_0\tau^*} = -\omega_0\tau_2$$

$$\Rightarrow A\left(\frac{\tau_1\tau_2\omega_0^2(1-\tau_1\tau_2\omega_0^2) - \omega_0^2\tau^{*2} - 1 + \tau_1\tau_2\omega_0^2}{\omega_0\tau^*}\right) = -\omega_0\tau_2$$

$$\Rightarrow A = \frac{-\tau_2\tau^*\omega_0^2}{\tau_1\tau_2\omega_0^2 - \tau_1^2\tau_2^2\omega_0^4 - 1 + \tau_1\tau_2\omega_0^2 - \omega_0^2\tau^{*2}} = \nearrow$$

$$= \frac{\tau_2 \tau^* \omega_0^2}{(1-\omega_0^2\tau_1\tau_2)^2 + \omega_0^2 \tau^{*2}} = \frac{N_A}{D_A}$$

$$B = \frac{N_A}{D_A} \frac{(1-\tau_1\tau_2\omega_0^2)}{\omega_0 \tau^*}$$

$$\Downarrow$$

$$\sqrt{A^2+B^2} = \sqrt{\frac{N_A^2}{D_A^2} + \frac{N_A^2}{D_A^2}\frac{(1-\tau_1\tau_2\omega_0^2)^2}{(\omega_0\tau^*)^2}} = \frac{N_A}{D_A\, \omega_0 \tau^*}\sqrt{D_A} =$$

$$= \frac{\tau_2\tau^*\omega_0^2}{\omega_0\tau^*\sqrt{D_A}} = \frac{\omega_0\tau_2}{\sqrt{(1-\tau_1\tau_2\omega_0^2)^2 + \omega_0^2\tau^{*2}}}$$

A.4)

$$\sqrt{A^2+B^2} = \frac{1}{\sqrt{2}} \Rightarrow 2\omega_0^2\tau_2^2 = (1-\tau_1\tau_2\omega_0^2)^2 + \omega_0^2\tau^{*2}$$

$$\Rightarrow 2\omega_0^2\tau_2^2 = 1 + \tau_1^2\tau_2^2\omega_0^4 - 2\tau_1\tau_2\omega_0^2 + \omega_0^2\tau^{*2}$$

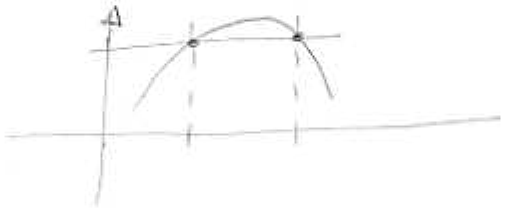
$\omega_0^2 = x$

$$\Rightarrow \underbrace{\tau_1\tau_2}_{a} x^2 - \underbrace{(2\tau_2^2 + 2\tau_1\tau_2 - \tau^{*2})}_{b} x + \underbrace{1}_{c} = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad b^2 - 4ac = (2\tau_2^2 + 2\tau_1\tau_2 - \tau^{*2})^2 - 4\tau_1\tau_2$$

if $b^2-4ac>0 \Rightarrow 2\tau_2^2+2\tau_1\tau_2-\tau^{*2}>2\sqrt{\tau_1\tau_2}$

then



we have two real solutions for x

and 4 solutions for ω_0 (only the positive solutions count).

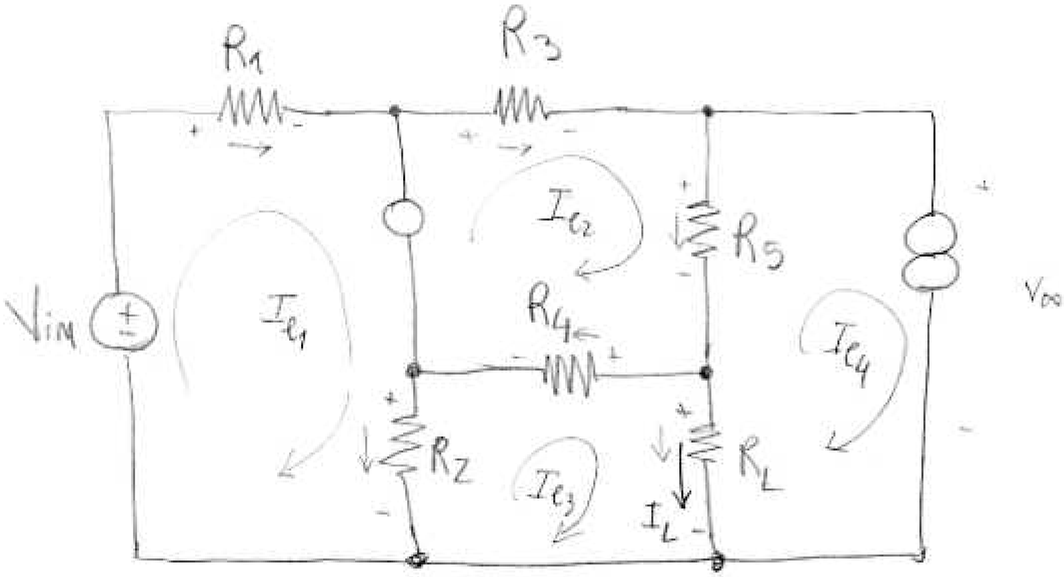
if $b^2-4ac<0 \Rightarrow 2\tau_2^2+2\tau_1\tau_2-\tau^{*2}<2\sqrt{\tau_1\tau_2}$

then there are no real solutions

if $b^2-4ac=0 \Rightarrow (2\tau_2^2+2\tau_1\tau_2-\tau^{*2})^2=4\tau_1\tau_2$

$\Rightarrow$ there is only one solution for ω_0

B) Let's use the pair nullator - norator



There are 4 meshes so we will have 4 equations plus 2 equations from the nullator.
The unknown are the four mesh currents, the voltage across the norator and the current through the norator.

$$\begin{cases} -V_{in} + V_{R_1} + 0 + V_{R_2} = 0 \\ 0 + V_{R_3} + V_{R_5} + V_{R_4} = 0 \\ -V_{R_2} - V_{R_4} + V_{R_L} = 0 \\ -V_{R_L} - V_{R_5} + V_{\infty} = 0 \\ I_{l_1} - I_{l_2} = 0 \end{cases} \quad \begin{cases} -V_{in} + R_1 I_{l_1} + R_2 (I_{l_1} - I_{l_3}) = 0 \\ R_3 I_{l_2} + R_5 (I_{l_2} - I_{l_4}) + R_4 (I_{l_2} - I_{l_3}) = 0 \\ R_2 (I_{l_3} - I_{l_1}) + R_4 (I_{l_3} - I_{l_2}) + R_L (I_{l_3} - I_{l_4}) = 0 \\ R_L (I_{l_4} - I_{l_3}) + R_5 (I_{l_4} - I_{l_2}) + V_{\infty} = 0 \\ I_{l_1} - I_{l_2} = 0 \end{cases}$$

Since $R_3 = R_4 + R_5$ and $R_1 = R_2$

$$\begin{cases} -V_{im} + 2R_1 I_{e_1} - R_1 I_{e_3} = 0 & 1 \\ 2R_3 I_{e_2} - R_5 I_{e_4} - R_4 I_{e_3} = 0 & 2 \\ (R_2 + R_4 + R_L) I_{e_3} - R_2 I_{e_1} - R_4 I_{e_2} - R_L I_{e_4} = 0 & 3 \\ I_{e_1} = I_{e_2} & 4 \end{cases}$$

We actually need only eq. 1, 2, 4 :

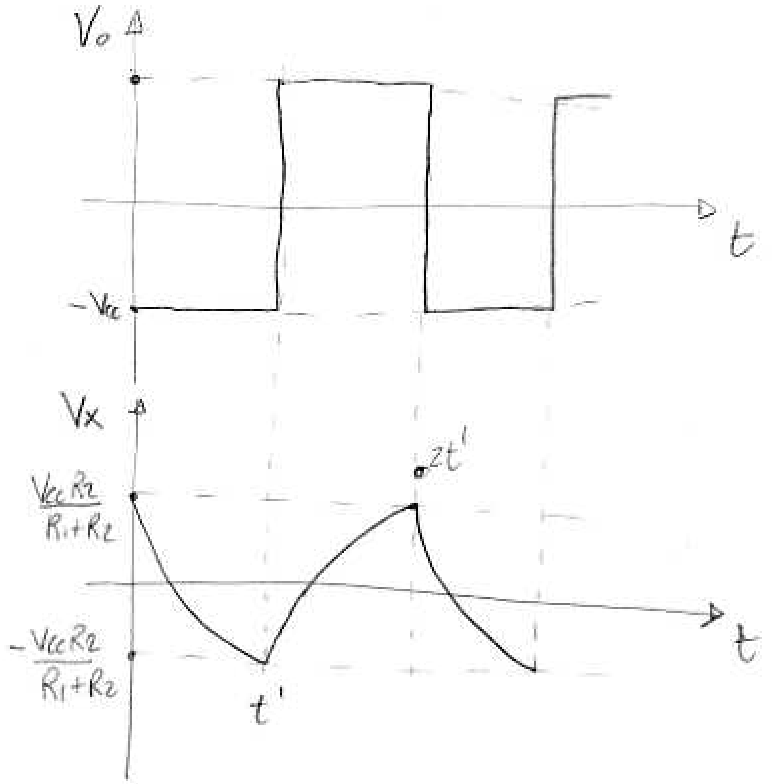
$$\begin{cases} -V_{im} + 2R_1 I_{e_1} - R_1 I_{e_3} = 0 \\ 2R_3 I_{e_1} - R_5 I_{e_4} - R_4 I_{e_3} = 0 \end{cases} \qquad \begin{cases} I_{e_3} = -\frac{V_{im}}{R_1} + 2I_{e_1} \\ I_{e_4} = \frac{2R_3}{R_5} I_{e_1} + \frac{V_{im} R_4}{R_1 R_5} - \frac{2R_4}{R_5} I_{e_1} \end{cases}$$

$$I_L = I_{e_3} - I_{e_4} = -\frac{V_{im}}{R_1} + 2I_{e_1} - \left(\frac{\overset{\downarrow 2R_5}{2R_3 - 2R_4}}{R_5} I_{e_1} + \frac{V_{im} R_4}{R_1 R_5} \right)$$

$$= -\frac{V_{im}}{R_1} + \cancel{2I_{e_1}} - \cancel{2I_{e_1}} - \frac{V_{im} R_4}{R_1 R_5} =$$

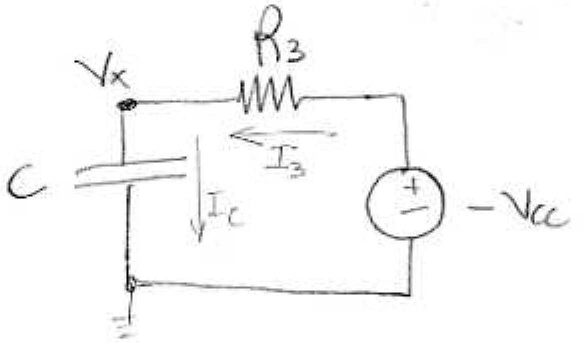
$$= -\frac{R_5 V_{im} + R_4 V_{im}}{R_1 R_5} = -\frac{V_{im} R_3}{R_1 R_5}$$

c)



We need t' and then $f = \frac{1}{2t'}$

the capacitor discharges on R_3 so the circuit is like



$v_c(0) = V_{cc} \frac{R_2}{R_1+R_2}$

We need an expression for V_x

$$I_3 - I_c = 0 \Rightarrow \frac{-V_{cc} - V_x}{R_3} - C\frac{dV_x}{dt} = 0$$

$$R_3C\frac{dV_x}{dt} + V_x = -V_{cc}$$

consider $V_x = Ae^{\alpha t} + B$ as our guess:

$$R_3CA\alpha e^{\alpha t} + Ae^{\alpha t} + B = -V_{cc}$$

$$B = -V_{cc}$$

$$R_3C\cancel{A}\alpha \cancel{e^{\alpha t}} + \cancel{Ae^{\alpha t}} = 0 \Rightarrow \alpha = -\frac{1}{R_3C}$$

$$V_x(0) = V_{cc}\frac{R_2}{R_1+R_2} \Rightarrow A - V_{cc} = V_{cc}\frac{R_2}{R_1+R_2}$$

$$\Rightarrow A = V_{cc}\left(\frac{R_2}{R_1+R_2} + 1\right)$$

Now we want to now the time when

$V_x = -V_{cc}\frac{R_2}{R_1+R_2}$:

$$-\frac{V_{cc}R_2}{R_1+R_2} = V_{cc}\left(\frac{R_2}{R_1+R_2} + 1\right)e^{-t'/R_3C} - V_{cc}$$

$$\cancel{V_{cc}}\left(1 - \frac{R_2}{R_1+R_2}\right) = \cancel{V_{cc}}\left(\frac{R_2}{R_1+R_2} + 1\right)e^{-t'/R_3C}$$

$$\frac{R_1}{\cancel{R_1+R_2}} = \frac{2R_2+R_1}{\cancel{R_1+R_2}} e^{-t'/R_3C}$$

$$e^{-t'/R_3C} = \frac{R_1}{2R_2+R_1} \Rightarrow -\frac{t'}{R_3C} = \ln\left(\frac{R_1}{2R_2+R_1}\right)$$

$$\Rightarrow t' = -R_3C \ln\left(\frac{R_1}{2R_2+R_1}\right)$$

$$f = \frac{1}{2t'} = -\frac{1}{2R_3C \ln\left(\frac{R_1}{2R_2+R_1}\right)}$$