Second Order Circuits

Lecture 36, 11/30/05
Corrected 12/1/05

OUTLINE

- 2nd order Differential Equation (α and ω_0)
- Mechanical Analogy (height = v; velocity = i)
- Solution to 2nd Order Equations
- Cases: Under, critical and over damping
- Example: Fig 4.21 Hambley

Reading

Hambley pp 167-175

2nd Order Systems Are Very Important

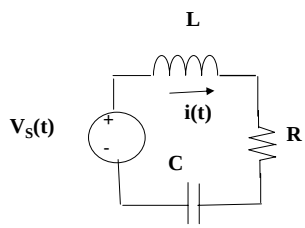
2nd order systems arise frequently in practice because there are often two alternate forms of energy storage that exchange energy.

- Electronic Circuits: The energy stored in the electric fields in capacitors can be exchanged with the energy stored in magnetic fields in inductors.
- Mechanical Systems: The energy stored in the gravitational potential of a mass due to its height can be exchanged with the kinetic energy due to the velocity of the mas.

Example 2nd Order Systems

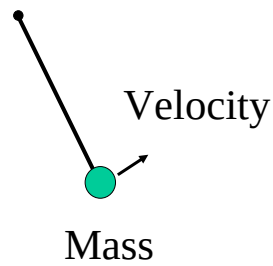
Electrical

RLC Circuit

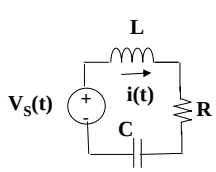


Mechanical

Pendulum



Differential Equation for RLC Series Circuit



Apply KVL,
Take derivative,
Divide by L

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt + V_C(0) = v_s(t)$$

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt}$$

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

2nd Order Equation

Standard Form of 2nd Order Equation

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)$$

$\alpha = \frac{R}{2L}$ Damping Coefficient

$\omega_0 = \frac{1}{\sqrt{LC}}$ Undamped Resonate Frequency

$\zeta = \frac{\alpha}{\omega}$ Damping Ratio

Solving a Nth Order Differential Equation

- Substitute trial solution Ke^{st}
- Produces an algebraic characteristic polynomial of order N
- Solution is summation of the trial solutions

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0$$

$$s^2 Ke^{st} + 2\alpha s Ke^{st} + \omega_0^2 Ke^{st} = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

Polynomial is of order N and has N roots s_i i = 1,N

$$i(t) = \sum_{i=1}^{N} K_i e^{s_i t}$$

Sum is over N

Watch out for equal roots.

Solution to a 2nd Order Differential Equations

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Damping

$$i_{Char}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$ Over (Exp)

$$i_{Char}(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$ Critical

$$i_{Char}(t) = K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t)$$ Under (Oscl.)

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2}$$

Example From Hambley Fig 4.21

- Vs = 10V step; L = 10 mH; C = 1 μF; $\omega_0 = 10^4$

$$i(t) = C\frac{dv_C(t)}{dt}$$

$$L\frac{di(t)}{dt} + Ri(t) + v_C(t) = V_S$$

$$LC\frac{d^2 v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_C(t) = V_S$$

$$\frac{d^2 v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = \frac{V_S}{LC}$$

$$v_{c\_P}(t) = V_S = 10V$$

Solution Case I (Overdamped)

Case I (R = 300 Ω)

$$v_{Char}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$\alpha = \frac{R}{2L} = 1.5 \cdot 10^4$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -0.3820 \cdot 10^4$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2.618 \cdot 10^4$$

$$v_{c\_TOTAL}(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

Initial Conditions: Case I (Overdamped)

2nd Order requires two Initial conditions

- Capacitor voltage (Given as zero)
- Inductor Current (Given as zero)

$$v_{c\_TOTAL}(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$v_{c\_TOTAL}(0) = 10 + K_1 + K_2$$

$$i_{L\_TOTAL}(0) = i_{C\_TOTAL}(0) = \frac{1}{C}\frac{dv_c(t)}{dt} = 0 \Rightarrow s_1 K_1 + s_2 K_2 = 0$$

$$v_{c\_TOTAL}(t) = 10 + 1.708 e^{s_1 t} - 11.708 K_2 e^{s_2 t}$$

See Fig. 4.23

Solution Case II (Critically Damped)

- R = 200 Ω

$$v_{Char}(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$
$$\alpha = \frac{R}{2L} = 10^4$$
$$s_1 = s_2 = -\alpha = -10^4$$
$$v_{c\_TOTAL}(t) = 10 + K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$
$$v_{c\_TOTAL}(0) = 0 => 10 + K_1 = 0$$
$$i_{L\_TOTAL}(0) = i_{C\_TOTAL}(0) = \frac{1}{C}\frac{dv_{c\_TOTAL}(t)}{dt} = 0 => s_1 K_1 + K_2 = 0$$
$$v_{c\_TOTAL}(t) = 10 - 10e^{s_1 t} - 10^5 t e^{s_1 t}$$

See Fig. 4.24

Solution Case III (Underdamped)

- R = 100 Ω

$$v_{Char}(t) = K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t)$$
$$\alpha = \frac{R}{2L} = 5000$$
$$\omega_n = \sqrt{\omega_o^2 - \alpha^2} = 8660$$
$$v_{C\_TOTAL}(t) = 10 + K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t)$$
$$v_{c\_TOTAL}(0) = 0 => 10 + K_1 = 0$$
$$i_{L\_TOTAL}(0) = i_{L\_TOTAL}(0) = \frac{1}{C}\frac{dv_{c\_TOTAL}(t)}{dt} = 0 => -\alpha K_1 + \omega_n K_2 = 0$$
$$v_{C\_TOTAL}(t) = 10 - 10e^{-\alpha t}\cos(\omega_n t) - 5.774e^{-\alpha t}\sin(\omega_n t)$$

See Fig. 4.25 and compare all 3 in Fig. 4.26