2nd Order Circuits

OUTLINE

- 2nd order Differential Equation ($\alpha$ and $\omega_0$)
- Mechanical Analogy (height = v; velocity = i)
- Solution to 2nd Order Equations
- Cases: Under, critical and over damping
- Example: Fig 4.21 Hambley

Reading
Hambley pp 167-175

2nd Order Systems Are Very Important

2nd order systems arise frequently in practice because there are often two alternate forms of energy storage that exchange energy.

- Electronic Circuits: The energy stored in the electric fields in capacitors can be exchanged with the energy stored in magnetic fields in inductors.
- Mechanical Systems: The energy stored in the gravitational potential of a mass due to its height can be exchanged with the kinetic energy due to the velocity of the mas.
Example 2\textsuperscript{nd} Order Systems

Electrical
RLC Circuit

Mechanical
Pendulum

\[ V_S(t) \]
\[ i(t) \]

\[ \begin{align*}
L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(t) dt + V_C(0) &= v_s(t) \\
L \frac{d^2i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C}i(t) &= \frac{dv_s(t)}{dt} \\
\frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC}i(t) &= \frac{1}{L} \frac{dv_s(t)}{dt}
\end{align*} \]

2\textsuperscript{nd} Order Equation
**Standard Form of 2nd Order Equation**

\[
\frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = \frac{1}{L} \frac{dv_s(t)}{dt}
\]

\[
\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)
\]

\[
\alpha = \frac{R}{2L} \quad \text{Damping Coefficient}
\]

\[
\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{Undamped Resonate Frequency}
\]

\[
\zeta = \frac{\alpha}{\omega} \quad \text{Damping Ratio}
\]

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**Solving a N\textsuperscript{th} Order Differential Equation**

- Substitute trial solution $Ke^{st}$
- Produces an algebraic characteristic polynomial of order $N$
- Solution is summation of the trial solutions

\[
\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]

\[
s^2 Ke^{st} + 2\alpha sKe^{st} + \omega_0^2 Ke^{st} = 0
\]

\[
s^2 + 2\alpha s + \omega_0^2 = 0 \quad \text{Polynomial is of order N}
\]

\[
i(t) = \sum_{i=1}^{N} K_i e^{s_i t} \quad \text{and has N roots } s_i \text{ i = 1,N}
\]

\[
i(t) = \sum_{i=1}^{N} K_i e^{s_i t} \quad \text{Sum is over N}
\]

Watch out for equal roots.
Solution to a 2nd Order Differential Equations

\[
\frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0
\]
\[
s^2 + 2\alpha s + \omega_0^2 = 0
\]
\[
s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}
\]
\[
s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}
\]

Damping

\[
i_{\text{Char}}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}
\]

Over (Exp)

\[
i_{\text{Char}}(t) = K_1 e^{s_1 t} + K_2 te^{s_1 t}
\]

Critical

\[
i_{\text{Char}}(t) = K_1 e^{\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]

Under (Oscil.)

\[
\omega_n = \sqrt{\omega_0^2 - \alpha^2}
\]

Example From Hambley Fig 4.21

- \( V_s = 10V \) step; \( L = 10 \text{ mH} \); \( C = 1 \mu F \); \( \omega_0 = 10^4 \)

\[
i(t) = C \frac{dv_c(t)}{dt}
\]

\[
L \frac{di(t)}{dt} + Ri(t) + v_c(t) = V_s
\]

\[
LC \frac{d^2v_c(t)}{dt^2} + RC \frac{dv_c(t)}{dt} + v_c(t) = V_s
\]

\[
\frac{d^2v_c(t)}{dt^2} + \frac{R}{L} \frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = \frac{V_s}{LC}
\]

\[v_{c\_p}(t) = V_s = 10V\]
Solution Case I (Overdamped)

Case I (R = 300 Ω)

\[ v_{\text{Char}}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \]
\[ \alpha = \frac{R}{2L} = 1.5 \cdot 10^4 \]
\[ s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -0.3820 \cdot 10^4 \]
\[ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -2.618 \cdot 10^4 \]
\[ v_{c_{\text{TOTAL}}}(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \]

Initial Conditions: Case I (Overdamped)

2nd Order requires two Initial conditions
- Capacitor voltage (Given as zero)
- Inductor Current (Given as zero)

\[ v_{c_{\text{TOTAL}}}(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \]
\[ v_{c_{\text{TOTAL}}}(0) = 10 + K_1 + K_2 \]
\[ i_{L_{\text{TOTAL}}}(0) = i_{c_{\text{TOTAL}}}(0) = \frac{1}{C} \frac{dv_c(t)}{dt} = 0 \Rightarrow s_1 K_1 + s_2 K_2 = 0 \]
\[ v_{c_{\text{TOTAL}}}(t) = 10 + 1.708 e^{s_1 t} - 11.708 K_2 e^{s_2 t} \]

See Fig. 4.23
Solution Case II (Critically Damped)

- $R = 200\ \Omega$

\[
v_{\text{ch}}(t) = K_1 e^{t/\tau} + K_2 e^{-t/\tau}
\]
\[
\alpha = \frac{R}{2L} = 10^4
\]
\[
s_1 = s_2 = -\alpha = -10^4
\]
\[
v_{c,\text{TOTAL}}(t) = 10 + K_1 e^{t/\tau} + K_2 e^{-t/\tau}
\]
\[
v_{c,\text{TOTAL}}(0) = 0 \Rightarrow 10 + K_1 = 0
\]
\[
i_{L,\text{TOTAL}}(0) = i_{c,\text{TOTAL}}(0) = \frac{1}{C} \frac{dv_{c,\text{TOTAL}}(t)}{dt} = 0 \Rightarrow s_1 K_1 + K_2 = 0
\]
\[
v_{c,\text{TOTAL}}(t) = 10 - 10 e^{t/\tau} - 10^4 t e^{-t/\tau}
\]

See Fig. 4.24

Solution Case III (Underdamped)

- $R = 100\ \Omega$

\[
v_{\text{ch}}(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]
\[
\alpha = \frac{R}{2L} = 5000
\]
\[
\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 8660
\]
\[
v_{c,\text{TOTAL}}(t) = 10 + K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t)
\]
\[
v_{c,\text{TOTAL}}(0) = 0 \Rightarrow 10 + K_1 = 0
\]
\[
i_{L,\text{TOTAL}}(0) = i_{c,\text{TOTAL}}(0) = \frac{1}{C} \frac{dv_{c,\text{TOTAL}}(t)}{dt} = 0 \Rightarrow -\alpha K_1 + \omega_n K_2 = 0
\]
\[
v_{c,\text{TOTAL}}(t) = 10 - 10 e^{-\alpha t} \cos(\omega_n t) - 5.774 e^{-\alpha t} \sin(\omega_n t)
\]

See Fig. 4.25 and compare all 3 in Fig. 4.26