## Filter Transfer Functions vs. Frequency

**Lecture 19, 10/12/05**

**OUTLINE**

- Transfer Function
- Common Filter Types
- First Order Lowpass and Highpass
- Bode Plots using notation of Hambley
- LC Filter Resonance

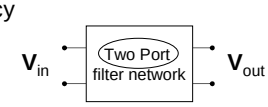
**Reading**

Hambley 6.4-6.7



## Transfer Function

- Transfer function is a function of frequency
  - Complex quantity
  - Both magnitude and phase are function of frequency




$$\mathbf{H}(f) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{out}}{V_{in}} \angle(\theta_{out} - \theta_{in})$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

Hambley Notation



## Filters

- Circuit designed to retain a certain frequency range and discard others

  *Low-pass*: pass low frequencies and reject high frequencies

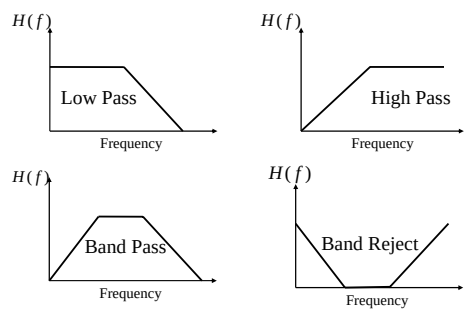
  *High-pass*: pass high frequencies and reject low frequencies

  *Band-pass*: pass some particular range of frequencies, reject other frequencies outside that band

  *Notch*: reject a range of frequencies and pass all other frequencies



## Common Filter Transfer Function vs. Freq






## First-Order Lowpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V_C}}{\mathbf{V}} = \frac{1/(j\omega C)}{1/(j\omega C) + R} = \frac{1}{1 + j\omega RC} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \angle -\tan^{-1}(\omega RC)$$

$$Let\ \omega_B = \frac{1}{RC} \quad and \quad f_B = \frac{1}{2\pi RC}$$

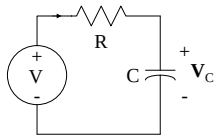
$$\mathbf{H(f)} = H(f)\angle\theta$$

Hambley Notation

$$H(f) = \frac{1}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}} \quad , \theta = -\tan^{-1}\left(\frac{f}{f_B}\right)$$

$$H(f_B) = \frac{1}{\sqrt{2}} = 2^{-1/2}$$

$$20\log_{10}\frac{H(f_B)}{H(0)} = 20(-\frac{1}{2})\log_{10} 2 = -3\ dB$$





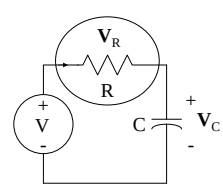

## First-Order Highpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V}_R}{\mathbf{V}} = \frac{R}{1/(j\omega C) + R} = \frac{j\omega RC}{1 + j\omega RC} = \frac{(\omega RC)}{\sqrt{1 + (\omega RC)^2}} \angle\left[\frac{\pi}{2} - \tan^{-1}(\omega RC)\right]$$

$$H(f) = \frac{\left(\frac{f}{f_B}\right)}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}} \quad , \theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{f_B}\right)$$

$$H(f_B) = \frac{1}{\sqrt{2}} = 2^{-1/2}$$

$$20\log_{10}\frac{H(f_B)}{H(0)} = 20(-\frac{1}{2})\log_{10} 2 = -3\ dB$$



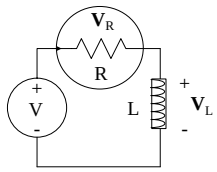
## First-Order Lowpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V}_B}{\mathbf{V}} = \frac{1}{\frac{j\omega L}{R}+1} = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \angle -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$Let\ \omega_B = \frac{R}{L} \quad and \quad f_B = \frac{R}{2\pi L}$$

$$\mathbf{H(f)} = H(f)\angle\theta$$

$$H(f) = \frac{1}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}} \quad ,\theta = -\tan^{-1}\left(\frac{f}{f_B}\right)$$





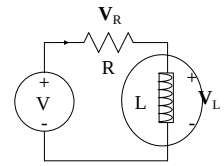

## First-Order Highpass Filter

$$\mathbf{H(f)} = \frac{\mathbf{V}_L}{\mathbf{V}} = \frac{\frac{j\omega L}{R}}{\frac{j\omega L}{R}+1} = \frac{\frac{\omega L}{R}}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \angle\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R}\right)\right]$$

$$Let\ \omega_B = \frac{R}{L} \quad and \quad f_B = \frac{R}{2\pi L}$$

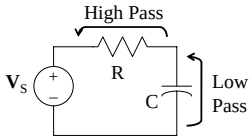
$$\mathbf{H(f)} = H(f)\angle\theta$$

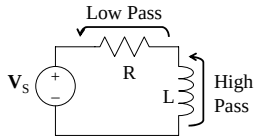
$$H(f) = \frac{\left(\frac{f}{f_B}\right)}{\sqrt{1+\left(\frac{f}{f_B}\right)^2}} \quad ,\theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{f_B}\right)$$






## First-Order Filter Circuits




$\mathbf{H}_R = R / (R + 1/j\omega C)$ $\qquad$ $\mathbf{H}_R = R / (R + j\omega L)$

$\mathbf{H}_C = (1/j\omega C) / (R + 1/j\omega C)$ $\qquad$ $\mathbf{H}_L = j\omega L / (R + j\omega L)$



## Change of Voltage or Current with A Change of Frequency

One may wish to specify the change of a quantity such as the output voltage of a filter when the frequency changes by a factor of 2 (an octave) or 10 (a decade).

For example, a single-stage RC low-pass filter has at frequencies above $\omega = 1/RC$ an output that changes at the rate -20dB per decade.



## Bode Plot

- Plot of magnitude of transfer function vs. frequency
  - Both x and y scale are in log scale
  - Y scale in dB
- Log Frequency Scale
  - Decade → Ratio of higher to lower frequency = 10
  - Octave → Ratio of higher to lower frequency = 2



## High-frequency asymptote of Lowpass filter

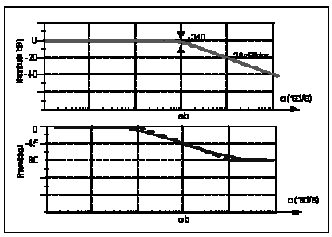
The high frequency asymptote of magnitude Bode plot assumes -20dB/decade slope

As $f \to \infty$

$$H(f) = \left(\frac{f}{f_B}\right)^{-1}$$

$$20\log_{10}\frac{H(10 f_B)}{H(f_B)} = -20dB$$

Hambley Notation



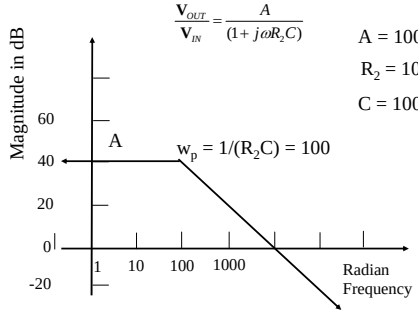
## Bode Plot: Label as dB

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{A}{(1 + j\omega R_2 C)}$$

A = 100

$R_2$ = 1000 Ohms

C = 100 uF




Note: Magnitude in dB = 20 $\log_{10}(V_{OUT}/V_{IN})$



## Low-frequency asymptote of Highpass filter

*As* $f \to 0$

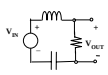
$$H(f) = \frac{\left(\frac{f}{f_B}\right)}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}} \xrightarrow[f \to \infty]{} \left(\frac{f}{f_B}\right)$$

$$20\log_{10}\frac{H(f_B)}{H(0.1 f_B)} = 20dB$$

The low frequency asymptote of magnitude Bode plot assumes 20dB/decade slope



## Series Resonance




Voltage divider

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{\mathbf{Z}_R}{\mathbf{Z}_L + \mathbf{Z}_R + \mathbf{Z}_C}$$

Substitute branch elements

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{R}{j\omega L + R + 1/j\omega C}$$

Arrange in resonance form

$$\frac{\mathbf{V}_{OUT}}{\mathbf{V}_{IN}} = \frac{R}{R + j(\omega L - 1/\omega C)}$$

Maximum when $w^2 = 1/(LC)$

Resonance quality factor

$$Q = \frac{\omega L}{R}$$

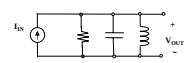
Ratio of reactance to resistance

Closely related to number of round trip cycles before 1/e decay.

Bandwidth is $f_0/Q$



## Parallel Resonance




Admittance

$$\mathbf{V}_{OUT} = \frac{\mathbf{I}_S}{\mathbf{Y}_L + \mathbf{Y}_R + \mathbf{Y}_C}$$

Substitute branch elements

$$\mathbf{V}_{OUT} = \frac{\mathbf{I}_S}{1/j\omega L + 1/R + jwC}$$

Arrange in resonance form

$$\mathbf{V}_{OUT} = \frac{\mathbf{I}_S}{1/R + j(\omega C - 1/\omega L)}$$

Maximum = $I_S/R$ when $w^2 = 1/(LC)$

Resonance quality factor

$$Q = \frac{\omega L}{R}$$

Ratio of reactance to resistance

Closely related to number of round trip cycles before 1/e decay.

Bandwidth is $f_0/Q$