RC and RL Circuits with General Sources

Lecture 13, 9/28/05

OUTLINE

• Quiz on Basics and Circuit Analysis
• Forced + Complementary = Total Solution
• Forced Response
• Adding Complementary to Meet Initial Cond.
• Example: Sinusoid
• Example: Arbitrary Forcing Function

Reading
Hambley 4.4

The Time Constant

• The complementary solution for any 1st order circuit is
  \[ x_c(t) = Ke^{-t/\tau} \]

• For an RC circuit, \( \tau = RC \)
• For an RL circuit, \( \tau = L/R \)

First Order Circuits

KVL around the loop:
\[ v_i(t) + v_c(t) = v_s(t) \]

RC \[ \frac{dv(t)}{dt} + v(t) = v_i(t) \]

KCL at the node:
\[ \frac{1}{C} \int v(x) dx = i_c(t) \]

\[ x(t) = \frac{1}{L} \int L \frac{di(t)}{dt} = i(t) \]

Complete Solution

• Voltages and currents in a 1st order circuit satisfy a differential equation of the form
  \[ x(t) + r \frac{dx(t)}{dt} = f(t) \]
  - \( f(t) \) is called the forcing function.
  - The complete solution is the sum of particular solution (forced response) and complementary solution (natural response).
  \[ x(t) = x_p(t) + x_c(t) \]
  - Particular solution satisfies the forcing function
  - Complementary solution is used to satisfy the initial conditions.
  - The initial condition determines the value of \( K \).
  \[ \frac{dx(t)}{dt} = 0 \] Homogeneous equation

\[ x_p(t) + \frac{dx(t)}{dt} = f(t) \]

\[ x_c(t) + \frac{dx(t)}{dt} = f(t) \]

\[ x(t) = Ke^{-t/\tau} \]

What Does \( X_c(t) \) Look Like?

• \( \tau \) is the amount of time necessary for an exponential to decay to 36.7% of its initial value.
• \(-1/\tau\) is the initial slope of an exponential with an initial value of 1.

The Particular Solution

• The particular solution \( x_p(t) \) is usually a weighted sum of \( f(t) \) and its first derivative.
  - If \( f(t) \) is constant, then \( x_p(t) \) is constant.
  - If \( f(t) \) is sinusoidal, then \( x_p(t) \) is sinusoidal.
The Particular Solution: F(t) Constant

\[ x_p(t) + t \frac{dx_p(t)}{dt} = F \]

Guess a solution
\[ x_p(t) = A + Bt \]

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero
\[ (B) = 0 \]
\[ (A + dB - F) = 0 \]
\[ B = 0 \]
\[ A = F \]

The Total Solution: F(t) Sinusoid

\[ x_p(t) + \frac{d^2x_p(t)}{dt^2} - F \sin(\omega t) + F_e \cos(\omega t) \]

\[ x_p(t) = A \sin(\omega t) + B \cos(\omega t) \]

Only K is unknown and is determined by the initial condition at \( t = 0 \)
\[ x_p(0) = A \sin(0) + B \cos(0) + K e^{\frac{\omega}{2}} = V_e \]
\[ x_p(t) = B + K = V_e \]

The Particular Solution: F(t) Sinusoid

\[ x_p(t) + t \frac{dx_p(t)}{dt} = F_e \sin(\omega t) + F \cos(\omega t) \]

Guess a solution
\[ x_p(t) = A \sin(\omega t) + B \cos(\omega t) \]

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero
\[ (A - dB - F) = 0 \]
\[ (B + dB + F_e) \cos(\omega t) = 0 \]
\[ A = F_e \]
\[ B + dB + F_e = 0 \]

The Particular Solution: F(t) Exp.

\[ x_p(t) + \frac{dx_p(t)}{dt} = F_e e^{-\alpha t} + F \]

Guess a solution
\[ x_p(t) = A + Be^{-\alpha t} \]

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero
\[ (B - \alpha t - F) = 0 \]
\[ (A + Be^{-\alpha t}) - \alpha t Be^{-\alpha t} = F_e e^{-\alpha t} + F \]
\[ (A - F) + (B - \alpha t - F_e) e^{-\alpha t} = 0 \]

Errors fixed 10/1/05