RC and RL Circuits with General Sources

Lecture 13, 9/28/05

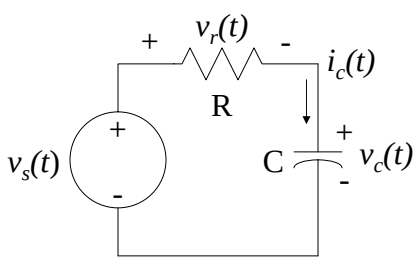
OUTLINE Errors fixed 10/1/05

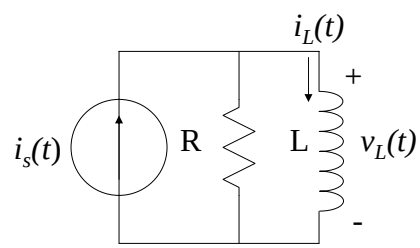
- Quiz on Basics and Circuit Analysis
- Forced + Complementary = Total Solution
- Forced Response
- Adding Complementary to Meet Initial Cond.
- Example: Sinusoid
- Example: Arbitrary Forcing Function

Reading

Hambley 4.4

First Order Circuits





KVL around the loop:

$v_r(t) + v_c(t) = v_s(t)$

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

KCL at the node:

$$\frac{v(t)}{R} + \frac{1}{L}\int_{-\infty}^{t} v(x)dx = i_s(t)$$

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

Complete Solution

- Voltages and currents in a 1st order circuit satisfy a differential equation of the form

$$x(t) + \tau \frac{dx(t)}{dt} = f(t)$$

  - *f(t)* is called the forcing function.
- The complete solution is the sum of particular solution (forced response) and complementary solution (natural response).

$$x(t) = x_p(t) + x_c(t)$$

  - Particular solution satisfies the forcing function
  - Complementary solution is used to satisfy the initial conditions.
  - The initial condition determines the value of *K.*

$$x_p(t) + \tau \frac{dx_p(t)}{dt} = f(t)$$

$$x_c(t) + \tau \frac{dx_c(t)}{dt} = 0$$

Homogeneous equation

$$x_c(t) = Ke^{-t/\tau}$$

The Time Constant

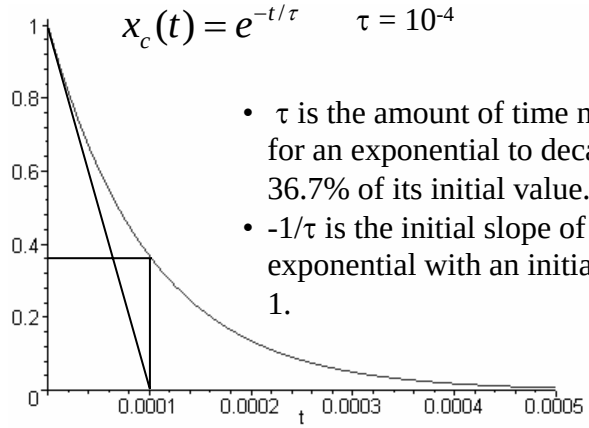
- The complementary solution for any 1st order circuit is

$$x_c(t) = Ke^{-t/\tau}$$

- For an RC circuit, $\tau = RC$
- For an RL circuit, $\tau = L/R$

What Does $X_c(t)$ Look Like?

$$x_c(t) = e^{-t/\tau} \qquad \tau = 10^{-4}$$



- τ is the amount of time necessary for an exponential to decay to 36.7% of its initial value.
- $-1/\tau$ is the initial slope of an exponential with an initial value of 1.

The Particular Solution

- The particular solution $x_p(t)$ is usually a weighted sum of $f(t)$ and its first derivative.
- If $f(t)$ is constant, then $x_p(t)$ is constant.
- If $f(t)$ is sinusoidal, then $x_p(t)$ is sinusoidal.

The Particular Solution: F(t) Constant

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F$$

Guess a solution

$$x_P(t) = A + Bt$$

$$(A + Bt) + \tau \frac{d(A + Bt)}{dt} = F$$

$$(A + Bt) + \tau B = F$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$(A + \tau B - F) + (B)t = 0$$

$$(B) = 0$$

$$B = 0$$

$$(A + \tau B - F) = 0$$

$$A = F$$

The Particular Solution: F(t) Sinusoid

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

Guess a solution

$$x_P(t) = A\sin(wt) + B\cos(wt)$$

$$(A\sin(wt) + B\cos(wt)) + \tau \frac{d(A\sin(wt) + B\cos(wt))}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$(A - \tau B - F_A)\sin(wt) + (B + \tau A - F_B)\cos(wt) = 0$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$(A - \tau B - F_A) = 0$$

$$(B + \tau A - F_B) = 0$$

Errors fixed 10/1/05

$$A = \frac{F_A + \tau F_B}{\tau^2 + 1} \qquad B = -\frac{\tau F_A - F_B}{\tau^2 + 1}$$

The Particular Solution: F(t) Exp.

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_1 e^{-\alpha t} + F_2$$

Guess a solution

$$x_P(t) = A + Be^{-\alpha t}$$

$$(A + Be^{-\alpha t}) + \tau \frac{d(A + Be^{-\alpha t})}{dt} = F_1 e^{-\alpha t} + F_2$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$(A + Be^{-\alpha t}) - \alpha \tau B e^{-\alpha t} = F_1 e^{-\alpha t} + F_2$$

$$(A - F_2) + (B - \alpha\tau - F_1)e^{-\alpha t} = 0$$

$$(B - \alpha\tau - F_1) = 0$$

$$B = \alpha\tau + F1$$

$$(A - F_2) = 0$$

$$A = F_2$$

The Total Solution: F(t) Sinusoid

$$x_P(t) + \tau \frac{dx_P(t)}{dt} = F_A \sin(wt) + F_B \cos(wt)$$

$$x_P(t) = A\sin(wt) + B\cos(wt)$$

$$A = \frac{F_A + \tau F_B}{\tau^2 + 1} \qquad B = -\frac{\tau F_A - F_B}{\tau^2 + 1}$$

$$x_C(t) = Ke^{-t/\tau}$$

Errors fixed 10/1/05

$$x_T(t) = A\sin(wt) + B\cos(wt) + Ke^{-t/\tau}$$

Only K is unknown and is determined by the initial condition at t =0

Example: $x_T(t=0) = V_C(t=0)$

$$x_T(0) = A\sin(0) + B\cos(0) + Ke^{-0/\tau} = V_C(t=0)$$

Errors fixed 10/1/05

$$x_T(0) = B + K = V_C(t=0) \qquad K = V_C(t=0) - B$$