RC and RL Circuits with General Sources

Lecture 13, 9/28/05

OUTLINE

• Quiz on Basics and Circuit Analysis
• Forced + Complementary = Total Solution
• Forced Response
• Adding Complementary to Meet Initial Cond.
• Example: Sinusoid
• Example: Arbitrary Forcing Function

Reading
Hambley 4.4

First Order Circuits

KVL around the loop:
\[ v_R(t) + v_c(t) = v_s(t) \]

\[ RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t) \]

KCL at the node:
\[ \frac{v(t)}{R} + \frac{1}{L} \int v(x) dx = i_s(t) \]

\[ \frac{L}{R} \frac{di_L(t)}{dt} + i_L(t) = i_s(t) \]
Complete Solution

- Voltages and currents in a 1st order circuit satisfy a differential equation of the form
  \[ x(t) + \tau \frac{dx(t)}{dt} = f(t) \]
  - \(f(t)\) is called the forcing function.
- The complete solution is the sum of particular solution (forced response) and complementary solution (natural response).
  \[ x(t) = x_p(t) + x_c(t) \]
  - Particular solution satisfies the forcing function
  - Complementary solution is used to satisfy the initial conditions.
  - The initial condition determines the value of \(K\).

\[
\begin{align*}
  x_p(t) + \tau \frac{dx_p(t)}{dt} &= f(t) \\
  x_c(t) + \tau \frac{dx_c(t)}{dt} &= 0 \\
  x_c(t) &= Ke^{-t/\tau}
\end{align*}
\]

The Time Constant

- The complementary solution for any 1st order circuit is
  \[ x_c(t) = Ke^{-t/\tau} \]
- For an RC circuit, \(\tau = RC\)
- For an RL circuit, \(\tau = L/R\)
What Does $X_c(t)$ Look Like?

- $\tau$ is the amount of time necessary for an exponential to decay to 36.7% of its initial value.
- $-1/\tau$ is the initial slope of an exponential with an initial value of 1.

\[ X_c(t) = e^{-t/\tau} \quad \tau = 10^{-4} \]

The Particular Solution

- The particular solution $x_p(t)$ is usually a weighted sum of $f(t)$ and its first derivative.
- If $f(t)$ is constant, then $x_p(t)$ is constant.
- If $f(t)$ is sinusoidal, then $x_p(t)$ is sinusoidal.
**The Particular Solution: F(t) Constant**

\[ x_p(t) + \tau \frac{dx_p(t)}{dt} = F \]

Guess a solution
\[ x_p(t) = A + Bt \]

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero
\[
\begin{align*}
B &= 0 \\
A &= F + \frac{\tau FA}{\tau^2 + 1}
\end{align*}
\]

**The Particular Solution: F(t) Sinusoid**

\[ x_p(t) + \tau \frac{dx_p(t)}{dt} = F_A \sin(\omega t) + F_B \cos(\omega t) \]

Guess a solution
\[ x_p(t) = A \sin(\omega t) + B \cos(\omega t) \]

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero
\[
\begin{align*}
(A - \tau B - F_A) \sin(\omega t) + (B + \tau A - F_B) \cos(\omega t) &= 0 \\
(\tau^2 + 1) A &= \frac{F_A + \tau F_B}{\tau^2 + 1} \\
B &= -\frac{\tau F_A - F_B}{\tau^2 + 1}
\end{align*}
\]

Errors fixed 10/1/05
The Particular Solution: $F(t)$ Exp.

Guess a solution

$$x_p(t) = A + Be^{-\alpha t}$$

$$x_p(t) + \tau \frac{dx_p(t)}{dt} = F_1 e^{-\alpha t} + F_2$$

Equation holds for all time and time variations are independent and thus each time variation coefficient is individually zero

$$(A + Be^{-\alpha t}) - \alpha \tau Be^{-\alpha t} = F_1 e^{-\alpha t} + F_2$$

$$(A - F_2) + (B - \alpha \tau - F_1) e^{-\alpha t} = 0$$

$$B = \alpha \tau + F_1$$

$$(A - F_2) = 0$$

The Total Solution: $F(t)$ Sinusoid

$$x_p(t) + \tau \frac{dx_p(t)}{dt} = F_1 \sin(\omega t) + F_B \cos(\omega t)$$

$$x_p(t) = A \sin(\omega t) + B \cos(\omega t)$$

$$x_c(t) = Ke^{-\frac{t}{\tau}}$$

$$x_f(t) = A \sin(\omega t) + B \cos(\omega t) + Ke^{-\frac{t}{\tau}}$$

Only $K$ is unknown and is determined by the initial condition at $t = 0$

Example: $x_f(t=0) = V_C(t=0)$

$$x_f(0) = A \sin(0) + B \cos(0) + Ke^{0/\tau} = V_C(t = 0)$$

$$x_f(0) = B + K = V_C(t = 0)$$

$$K = V_C(t = 0) - B$$