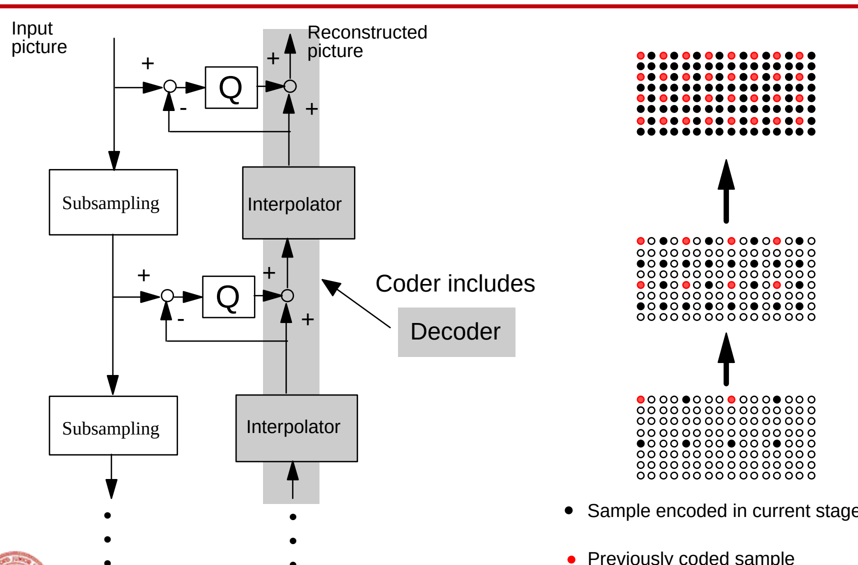
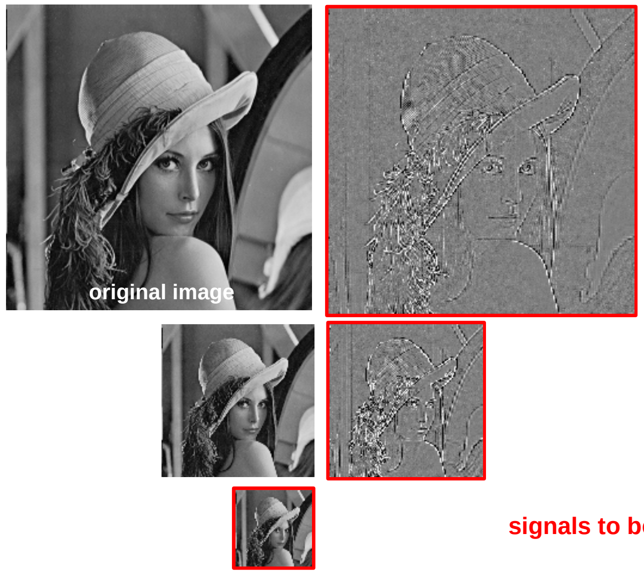
Multiresolution coding and wavelets

- Predictive (closed-loop) pyramids
- Open-loop ("Laplacian") pyramids
- Discrete Wavelet Transform (DWT)
- Quadrature mirror filters and conjugate quadrature filters
- Lifting and reversible wavelet transform
- Wavelet theory
- Embedded zero-tree wavelet (EZW) coding

Interpolation error coding, I

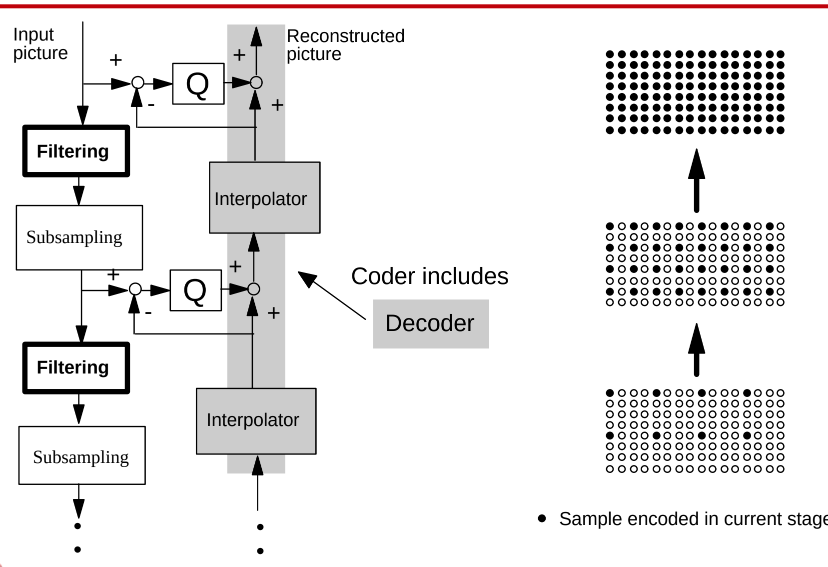


Interpolation error coding, II



signals to be encoded

Predictive pyramid, I



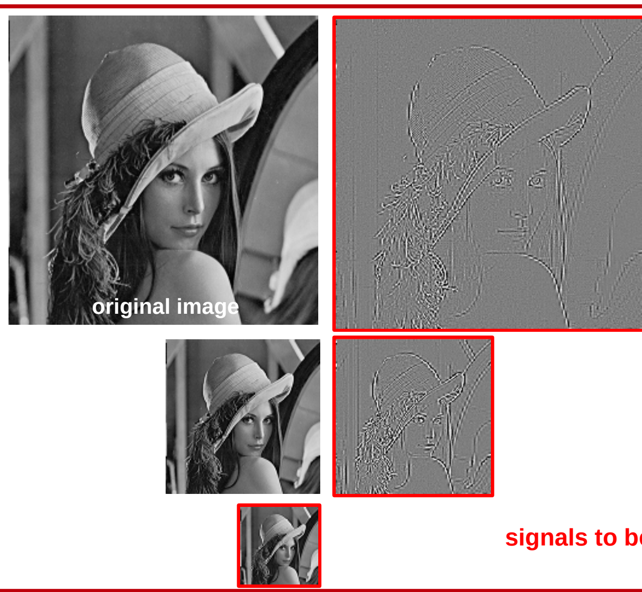
Predictive pyramid, II

Number of samples to be encoded =

$$\left(1+\frac{1}{N}+\frac{1}{N^2}+\ldots\right)=\frac{N}{N-1}$$ x number of original image samples

subsampling factor

Predictive pyramid, III



signals to be encoded

Comparison: interpolation error coding vs. pyramid

- Resolution layer #0, interpolated to original size for display

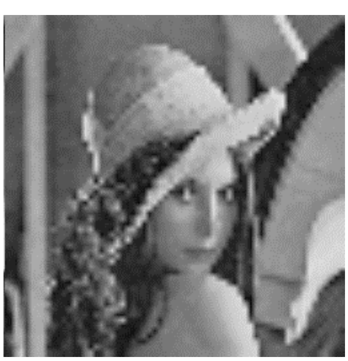
Interpolation Error Coding

Pyramid

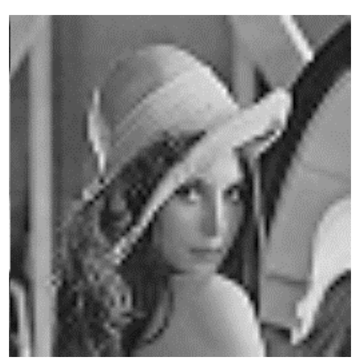
Comparison: interpolation error coding vs. pyramid

- Resolution layer #1, interpolated to original size for display

Interpolation Error Coding



Pyramid



Comparison: interpolation error coding vs. pyramid

- Resolution layer #2, interpolated to original size for display

<u>Interpolation Error Coding</u>

<u>Pyramid</u>

Comparison: interpolation error coding vs. pyramid

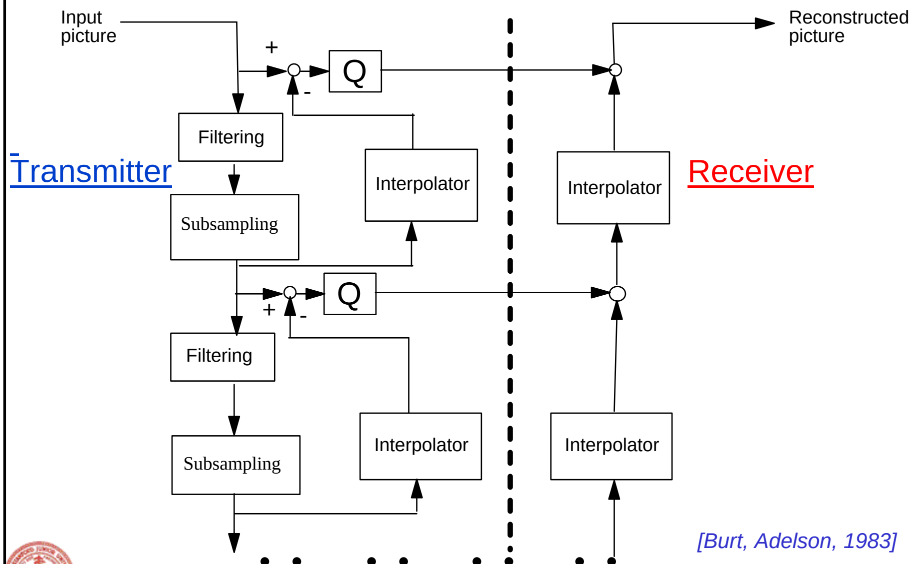
- Resolution layer #3

<u>Interpolation Error Coding</u>

=
(original)

<u>Pyramid</u>

Open-loop pyramid (Laplacian pyramid)

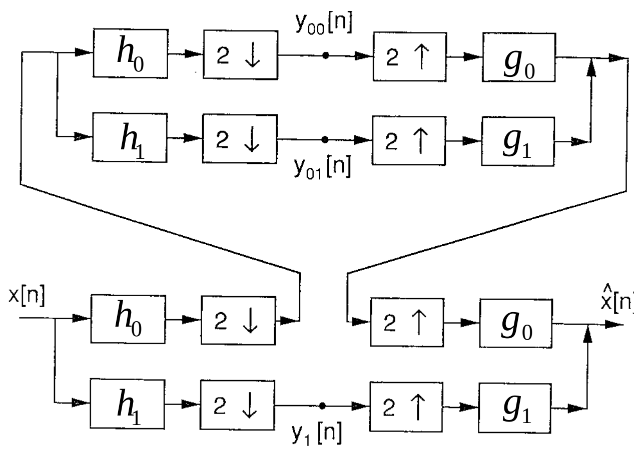


When multiresolution coding was a new idea . . .

This manuscript is okay if compared to some of the weaker papers. [. . .] however, I doubt that anyone will ever use this algorithm again.

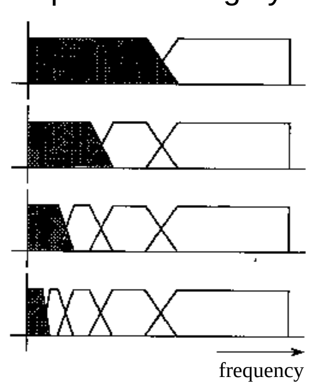
Anonymous reviewer of Burt and Adelson's original paper, ca. 1982

Cascaded analysis / synthesis filterbanks



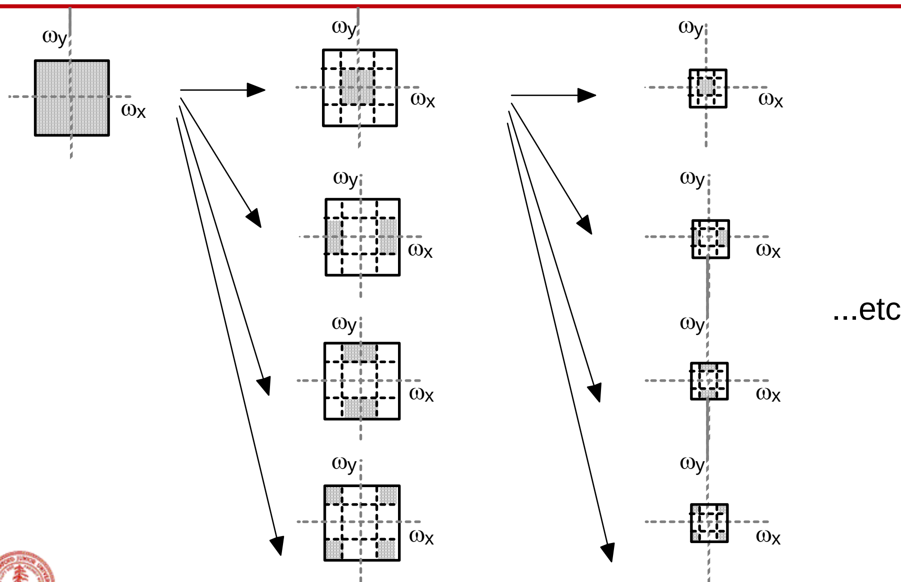
Discrete Wavelet Transform

- Recursive application of a two-band filter bank to the lowpass band of the previous stage yields octave band splitting:



- Same concept can be derived from wavelet theory: Discrete Wavelet Transform (DWT)

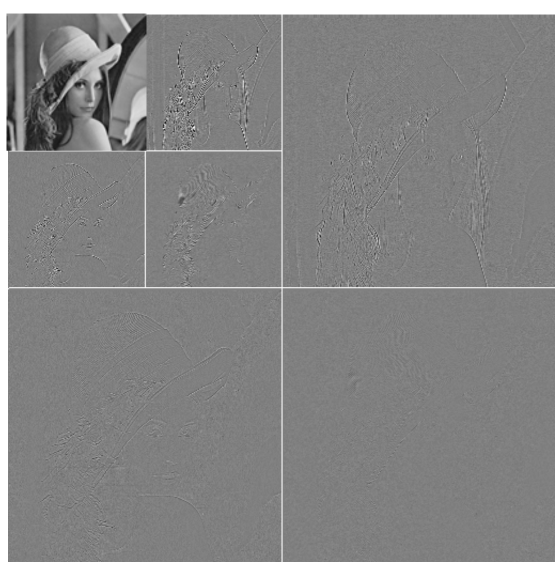
2-d Discrete Wavelet Transform



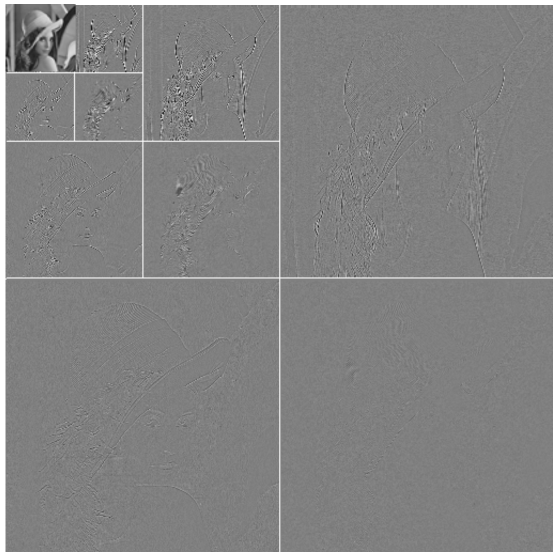
2-d Discrete Wavelet Transform example

2-d Discrete Wavelet Transform example

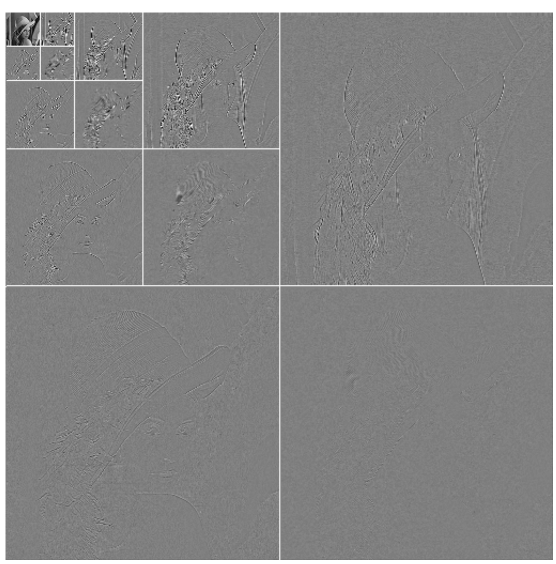
2-d Discrete Wavelet Transform example



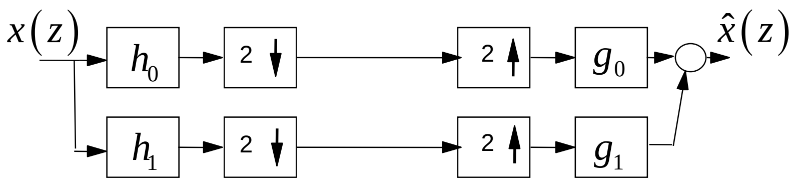
2-d Discrete Wavelet Transform example



2-d Discrete Wavelet Transform example



Two-channel filterbank



$$\hat{x}(z) = \frac{1}{2}\left[h_0(z)g_0(z) + h_1(z)g_1(z)\right]x(z)$$
$$+\frac{1}{2}\left[h_0(-z)g_0(z) + h_1(-z)g_1(z)\right]x(-z) \quad \text{Aliasing}$$

- Aliasing cancellation if :

$$g_0(z) = h_1(-z)$$
$$-g_1(z) = h_0(-z)$$

Example: two-channel filter bank with perfect reconstruction

- Impulse responses, analysis filters:

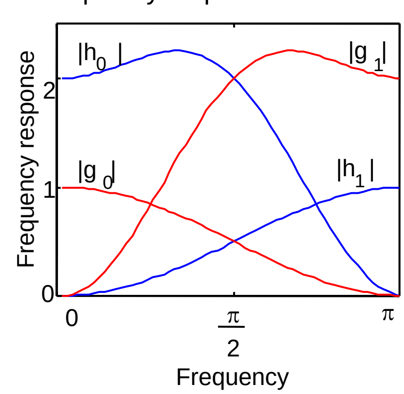
Lowpass $\left(\frac{-1}{4}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{4}\right)$ Highpass $\left(\frac{1}{4}, \frac{-1}{2}, \frac{1}{4}\right)$

- Impulse responses, synthesis filters

Lowpass $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ Highpass $\left(\frac{1}{4}, \frac{1}{2}, \frac{-3}{2}, \frac{1}{2}, \frac{1}{4}\right)$

"Biorthogonal 5/3 filters"
"LeGall filters"

- Mandatory in JPEG2000
- Frequency responses:



Classical quadrature mirror filters (QMF)

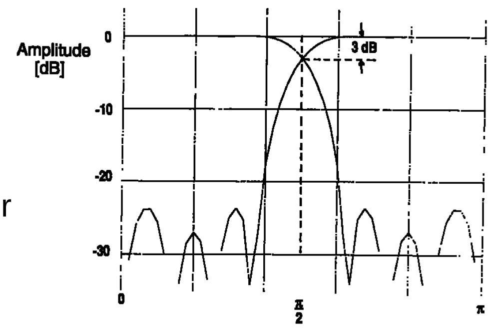
- QMFs achieve aliasing cancellation by choosing

$$h_1(z) = h_0(-z) = -g_1(z) = g_0(-z)$$

[Croisier, Esteban, Galand, 1976]

- Highpass band is the mirror image of the lowpass band in the frequency domain
- Need to design only one prototype filter

Example:
16-tap QMF filterbank



Conjugate quadrature filters

- Achieve aliasing cancelation by

Prototype filter

$$h_0(z) = g_0(z^{-1}) \equiv f(z)$$
$$h_1(z) = g_1(z^{-1}) = zf\left(-z^{-1}\right)$$

[Smith, Barnwell, 1986]

- Impulse responses

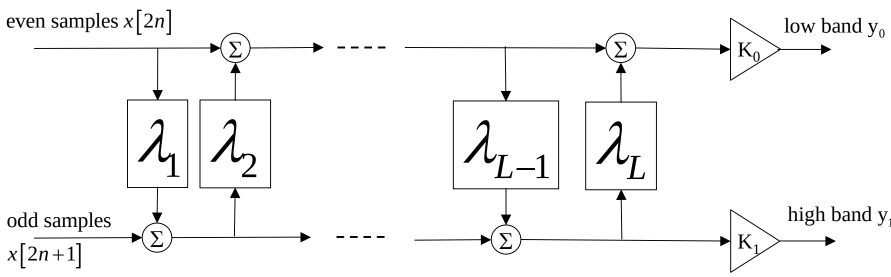
$$h_0[k] = g_0[-k] = f[k]$$
$$h_1[k] = g_1[-k] = (-1)^{k+1} f[-(k+1)]$$

- Orthonormal subband transform!
- Perfect reconstruction: find power complementary prototype filter

$$|F(\omega)|^2 + |F(\omega \pm \pi)|^2 = 2$$

Lifting

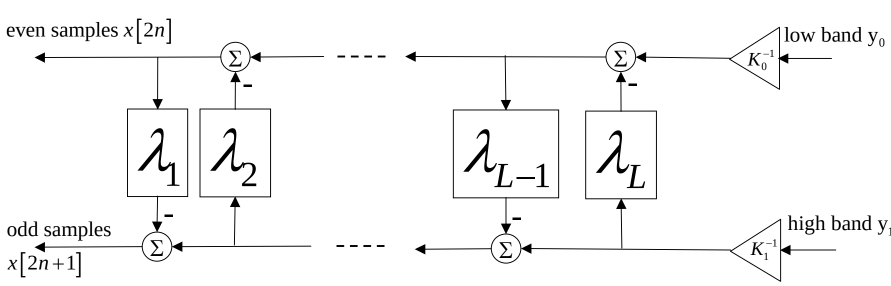
- Analysis filters



- L “lifting steps” *[Sweldens 1996]*
- First step can be interpreted as prediction of odd samples from the even samples

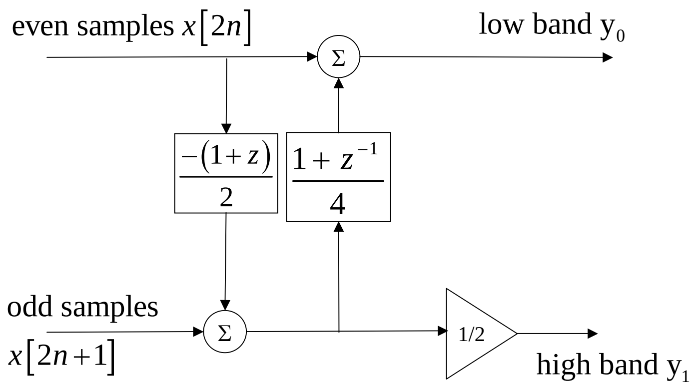
Lifting (cont.)

- Synthesis filters



- Perfect reconstruction (biorthogonality) is directly built into lifting structure
- Powerful for both implementation and filter/wavelet design

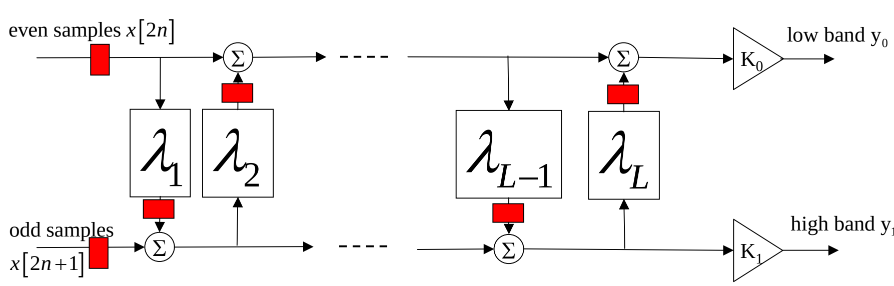
Example: lifting implementation of 5/3 filters



Verify by considering response to unit impulse in even and odd input channel.

Reversible subband transform

- Observation: lifting operators can be nonlinear.
- Incorporate the necessary rounding into lifting operator:



- Used in JPEG2000 as part of 5/3 biorthogonal wavelet transform

Wavelet bases

Consider Hilbert space $\mathcal{L}^2(\mathbb{R})$ of finite-energy functions $\mathbf{x} = x(t)$.

Wavelet basis for $\mathcal{L}^2(\mathbb{R})$: family of linearly independent functions

$$\psi_n^{(m)}(t) = \sqrt{2^{-m}}\,\psi\left(2^{-m}t - n\right)$$

"mother wavelet"

that span $\mathcal{L}^2(\mathbb{R})$. Hence any signal $\mathbf{x} \in \mathcal{L}^2(\mathbb{R})$ can be written as

$$\mathbf{x} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y^{(m)}[n]\, \psi_n^{(m)}$$

Multi-resolution analysis

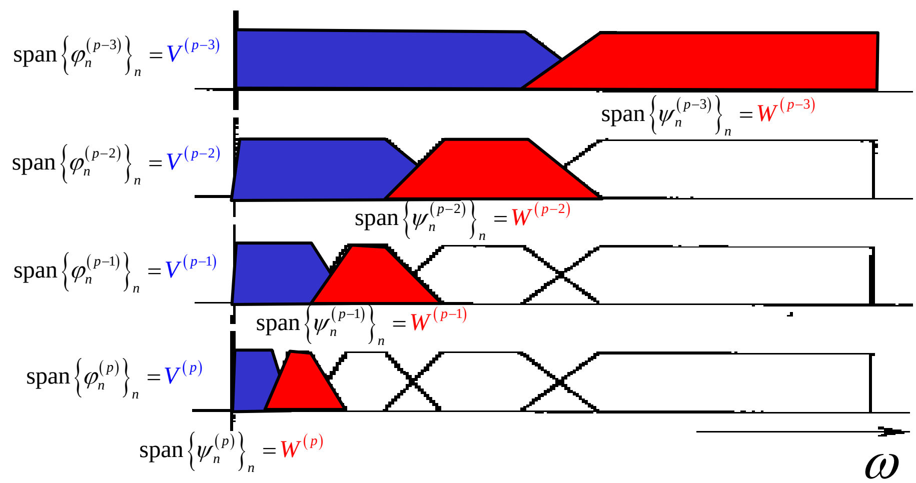
Nested subspaces

$$\ldots \subset V^{(2)} \subset V^{(1)} \subset V^{(0)} \subset V^{(-1)} \subset V^{(-2)} \subset \ldots \subset \mathcal{L}^2(\mathbb{R})$$

Upward completeness	$\bigcup_{m\in\mathbb{Z}} V^{(m)} = \mathcal{L}^2(\mathbb{R})$
Downward completeness	$\bigcap_{m\in\mathbb{Z}} V^{(m)} = \{\mathbf{0}\}$
Self-similarity	$x(t) \in V^{(0)}$ iff $x(2^{-m}t) \in V^{(m)}$
Translation invariance	$x(t) \in V^{(0)}$ iff $x(t-n) \in V^{(0)}$ for all $n \in \mathbb{Z}$

There exists a "scaling function" $\varphi(t)$ with integer translates $\varphi_n(t) = \varphi(t - n)$ such that $\{\varphi_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $V^{(0)}$

Multiresolution Fourier analysis



Relation to subband filters

Since $V^{(0)} \subset V^{(-1)}$, recursive definition of scaling function

$$\varphi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_0[n]\varphi_n^{(-1)}(t)}_{\substack{\text{linear combination} \\ \text{of scaling functions in } V^{(-1)}}} = \sqrt{2}\sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n)$$

Orthonormality

$$\delta[\mathrm{n}] = \left\langle \varphi_0^{(0)}, \varphi_n^{(0)} \right\rangle$$

$$= \int_{-\infty}^{\infty} \left(\sum_i g_0[i]\varphi_i^{(-1)}(t) \sum_j g_0[j]\varphi_{j+2n}^{(-1)}(t) \right) dt$$

$$= \sum_{i,j} g_0[i] g_0[j-2n] \left\langle \varphi_i^{(-1)}, \varphi_j^{(-1)} \right\rangle = \underbrace{\sum_i g_0[i] g_0[i-2n]}_{\substack{g_0[\mathrm{k}] \text{ unit norm and orthogonal} \\ \text{to its 2-translates: corresponds} \\ \text{to synthesis lowpass filter of} \\ \text{orthonormal subband transform}}}$$

Wavelets from scaling functions

$W^{(p)}$ is orthogonal complement of $V^{(p)}$ in $V^{(p-1)}$

$$W^{(p)} \perp V^{(p)} \quad \text{and} \quad W^{(p)} \cup V^{(p)} = V^{(p-1)}$$

Orthonormal wavelet basis $\left\{\psi_n^{(0)}\right\}$ for $W^{(0)} \subset V^{(-1)}$

$$\psi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_1[n]\varphi_n^{(-1)}(t)}_{\substack{\text{linear combination} \\ \text{of scaling functions in } V^{(-1)}}} = \sqrt{2}\sum_{n=-\infty}^{\infty} g_1[n]\varphi_n(2t-n)$$

Using conjugate quadrature high-pass synthesis filter

$$g_1[n] = (-1)^{n+1} g_0\left[-(n-1)\right]$$

The mutually orthonormal functions, $\left\{\psi_n^{(0)}\right\}_{n\in\mathbb{Z}}$ and $\left\{\varphi_n^{(0)}\right\}_{n\in\mathbb{Z}}$, together span $V^{(-1)}$.

Easy to extend to dilated versions of $\psi(t)$ to construct orthonormal wavelet basis

$$\left\{\psi_n^{(m)}\right\}_{n,m\in\mathbb{Z}} \text{ for } \mathcal{L}^2(\mathbb{R}).$$

Calculating wavelet coefficients for a continuous signal

- Signal synthesis by discrete filter bank

Suppose continuous signal $x^{(0)}(t) = \sum_{n\in\mathbb{Z}} y_0^{(0)}[n]\ \varphi(t-n) = \sum_{n\in\mathbb{Z}} y_0^{(0)}[n]\ \varphi_n^{(0)} \in V^{(0)}$

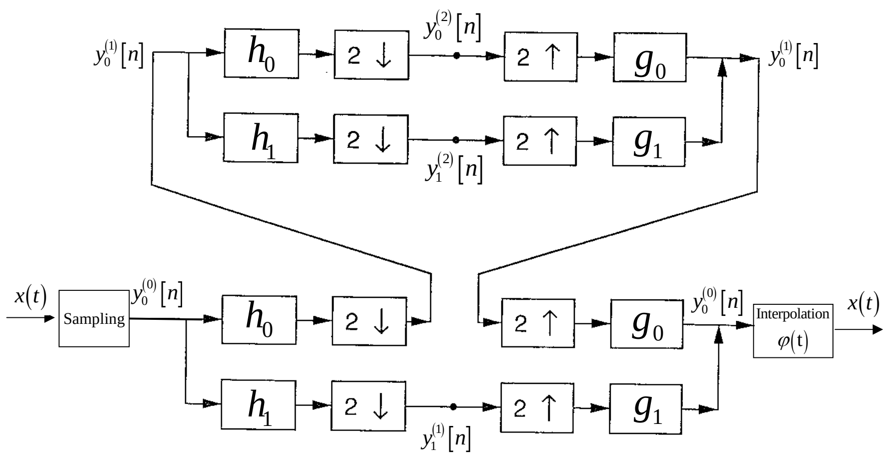
Write as superposition of $x^{(1)}(t) \in V^{(1)}$ and $w^{(1)}(t) \in W^{(1)}$

$$x^{(0)}(t) = \underbrace{\sum_{i\in\mathbb{Z}} y_0^{(1)}[i]\ \varphi_n^{(1)}}_{x^{(1)}(t)\in V^{(1)}} + \underbrace{\sum_{j\in\mathbb{Z}} y_1^{(1)}[j]\ \psi_n^{(1)}}_{w^{(1)}(t)\in W^{(1)}}$$

$$= \sum_{n\in\mathbb{Z}} \varphi_n^{(0)} \underbrace{\left(\sum_{i\in\mathbb{Z}} y_0^{(1)}[n]\, g_0[n-2i] + \sum_{j\in\mathbb{Z}} y_1^{(1)}[j]\, g_1[n-2i]\right)}_{y_0^{(0)}[n]}$$

- Signal analysis by analysis filters $h_0[k]$, $h_1[k]$
- Discrete wavelet transform

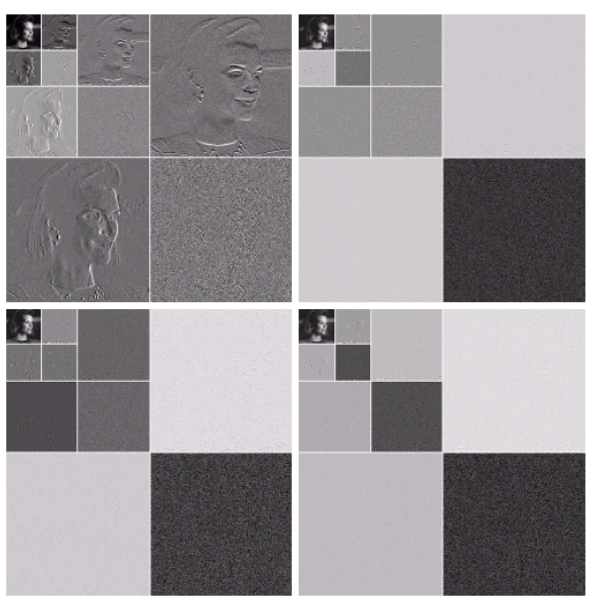
Discrete Wavelet Transform



Different wavelets

Haar
2/2 coeffs.

Daubechies
8/8

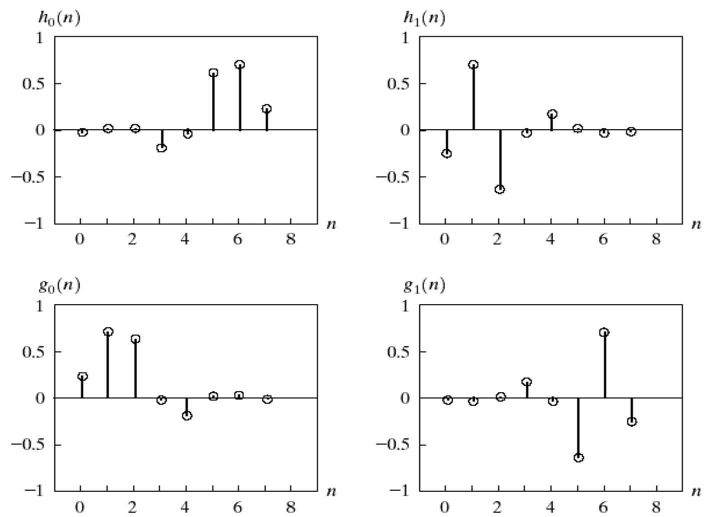


Symlets
8/8

Cohen-
Daubechies-
Feauveau
17/11

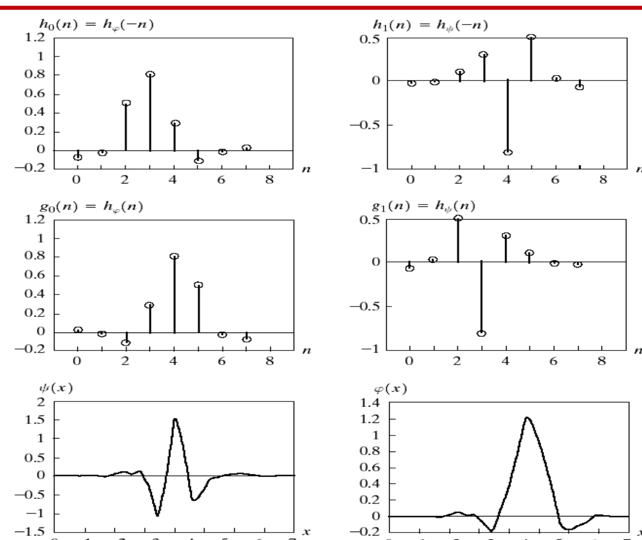
[Gonzalez, Woods, 2001]

Daubechies orthonormal 8-tap filters



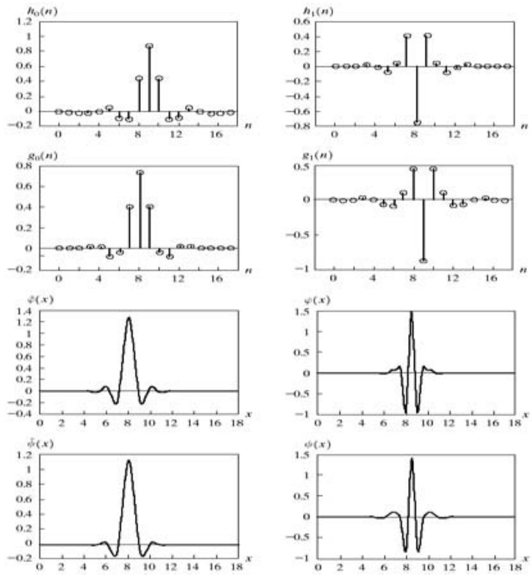
[Gonzalez, Woods, 2001]

8-tap Symlets



[Gonzalez, Woods, 2001]

Biorthogonal Cohen-Daubechies-Feauveau 17/11 wavelets



[Gonzalez, Woods, 2001]

Wavelet compression results

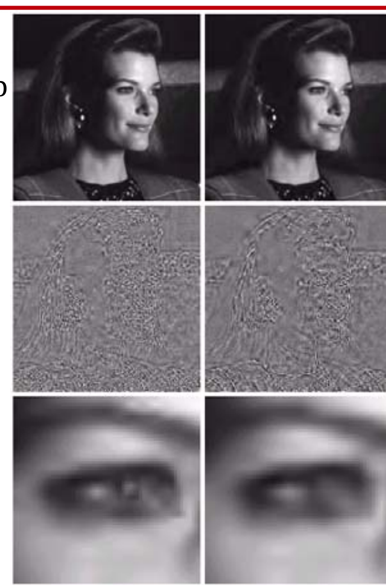
Original
512x512
8bpp

0.074 bpp

0.048 bpp

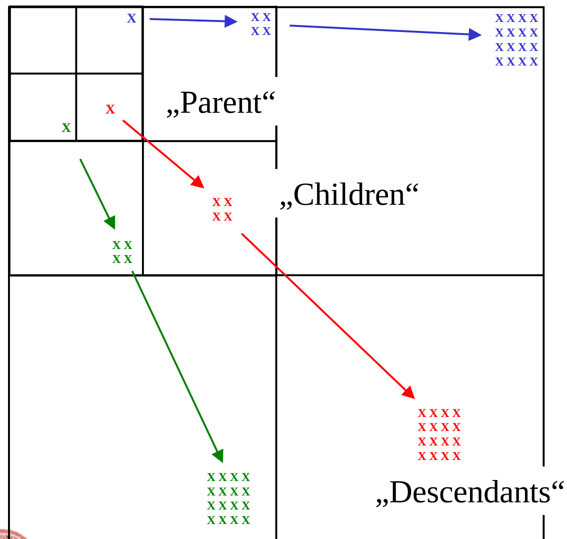
Error images

enlarged



[Gonzalez, Woods, 2001]

Embedded zero-tree wavelet algorithm



- Idea: Conditional coding of all descendants (incl. children)
- Coefficient magnitude > threshold: significant coefficients
- Four cases
  - ZTR: zero-tree, coefficient and all descendants are **not** significant
  - IZ: coefficient is not significant, but some descendants are significant
  - POS: positive significant
  - NEG: negative significant

Embedded zero-tree wavelet algorithm (cont.)

- For the highest bands, ZTR and IZ symbols are merged into one symbol Z
- Successive approximation quantization and encoding
  - Initial „dominant“ pass
    - Set initial threshold T, determine significant coefficients
    - Arithmetic coding of symbols ZTR, IZ, POS, NEG
  - Subordinate pass
    - Refine magnitude of **all** coefficients **found significant so far** by one bit (subdivide magnitude bin by two)
    - Arithmetic coding of sequence of zeros and ones.
  - Repeat dominant pass
    - Omit previously found significant coefficients
    - Decrease threshold by factor of 2, determine new significant coefficients
    - Arithmetic coding of symbols ZTR, IZ, POS, NEG
  - Repeat subordinate and dominate passes, until bit budget is exhausted.

Embedded zero-tree wavelet algorithm (cont.)

- Decoding: bitstream can be truncated to yield a coarser approximation: „embedded“ representation
- Further details: *J. M. Shapiro, „Embedded image coding using zerotrees of wavelet coefficients,“ IEEE Transactions on Signal Processing, vol. 41, no. 12, pp. 3445-3462, December 1993.*
- Enhancement SPIHT coder: *A. Said, A., W. A. Pearlman, „A new, fast, and efficient image codec based on set partitioning in hierarchical trees,“ IEEE Transactions on Circuits and Systems for Video Technology, vol. 63 , pp. 243-250, June 1996.*

Summary: multiresolution and subband coding

- Resolution pyramids with subsampling 2:1 horizontally and vertically
- Predictive pyramids: quantization error feedback („closed loop“)
- Transform pyramids: no quantization error feedback („open loop“)
- Pyramids: overcomplete representation of the image
- Critically sampled subband decomposition: number of samples not increased
- Discrete Wavelet Transform = cascaded dyadic subband splits
- Quadrature mirror filters and conjugate quadrature filters: aliasing cancellation
- Lifting: powerful for implementation and wavelet construction
- Lifting allows reversible wavelet transform
- Zero-trees: exploit statistical dependencies across subbands