Parameter estimation
class 5

Multiple View Geometry

Comp 290-089

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Content

- **Background**: Projective geometry (2D, 3D), Parameter estimation, Algorithm evaluation.
- **Single View**: Camera model, Calibration, Single View Geometry.
- **Two Views**: Epipolar Geometry, 3D reconstruction, Computing F, Computing structure, Plane and homographies.
- **Three Views**: Trifocal Tensor, Computing T.
- **More Views**: N-Linearities, Multiple view reconstruction, Bundle adjustment, auto-calibration, Dynamic SfM, Cheirality, Duality

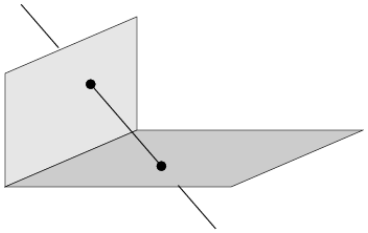
Multiple View Geometry course schedule

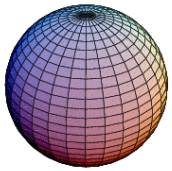
(subject to change)

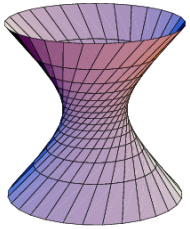
Jan. 7, 9	Intro & motivation	Projective 2D Geometry
Jan. 14, 16	(no class)	Projective 2D Geometry
Jan. 21, 23	Projective 3D Geometry	(no class)
Jan. 28, 30	Parameter Estimation	Parameter Estimation
Feb. 4, 6	Algorithm Evaluation	Camera Models
Feb. 11, 13	Camera Calibration	Single View Geometry
Feb. 18, 20	Epipolar Geometry	3D reconstruction
Feb. 25, 27	Fund. Matrix Comp.	Structure Comp.
Mar. 4, 6	Planes & Homographies	Trifocal Tensor
Mar. 18, 20	Three View Reconstruction	Multiple View Geometry
Mar. 25, 27	MultipleView Reconstruction	Bundle adjustment
Apr. 1, 3	Auto-Calibration	Papers
Apr. 8, 10	Dynamic SfM	Papers
Apr. 15, 17	Cheirality	Papers
Apr. 22, 24	Duality	Project Demos

Projective 3D Geometry

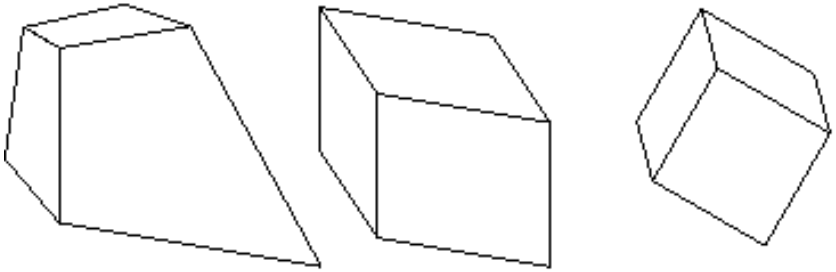
- Points, lines, planes and quadrics



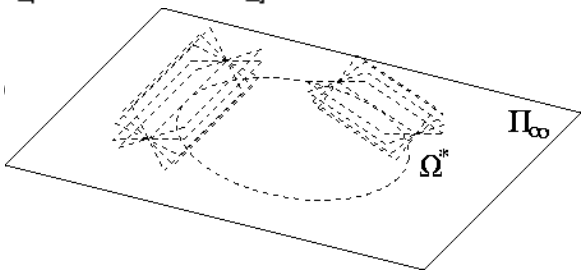




- Transformations

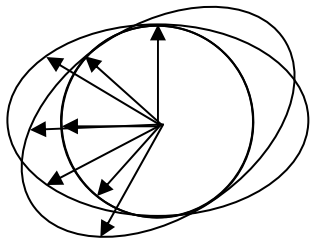


- Π_∞, ω_∞ and Ω



Singular Value Decomposition

$$A = U\Sigma V^T$$



$$U\Sigma V^T \mathbf{X}$$

Homogeneous least-squares

$$\min \|AX\| \text{ subject to } \|X\| = 1 \qquad \text{solution } X = V_n$$

Parameter estimation

- 2D homography

  Given a set of (x_i, x_i'), compute H $(x_i' = Hx_i)$
- 3D to 2D camera projection

  Given a set of (X_i, x_i), compute P $(x_i = PX_i)$
- Fundamental matrix

  Given a set of (x_i, x_i'), compute F $(x_i'^T F x_i = 0)$
- Trifocal tensor

  Given a set of (x_i, x_i', x_i''), compute T

Number of measurements required

- At least as many independent equations as degrees of freedom required
- Example: $\mathrm{x}' = \mathrm{H}\mathrm{x}$

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2 independent equations / point
8 degrees of freedom

4x2≥8

Approximate solutions

- Minimal solution

  4 points yield an exact solution for H
- More points
  - No exact solution, because measurements are inexact ("noise")
  - Search for "best" according to some cost function
  - Algebraic or geometric/statistical cost

Gold Standard algorithm

- Cost function that is optimal for some assumptions
- Computational algorithm that minimizes it is called "Gold Standard" algorithm
- Other algorithms can then be compared to it

Direct Linear Transformation (DLT)

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = 0 \qquad \mathbf{x}'_i = (x'_i, y'_i, w'_i)^\mathsf{T} \qquad \mathbf{H}\mathbf{x}_i = \begin{pmatrix} \mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \end{pmatrix}$$

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \begin{pmatrix} y'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i - w'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i - x'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i - y'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \end{pmatrix}$$

$$\begin{bmatrix} 0^\mathsf{T} & -w'_i\mathbf{x}_i^\mathsf{T} & y'_i\mathbf{x}_i^\mathsf{T} \\ w'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} & -x'_i\mathbf{x}_i^\mathsf{T} \\ -y'_i\mathbf{x}_i^\mathsf{T} & x'_i\mathbf{x}_i^\mathsf{T} & 0^\mathsf{T} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

$$\mathbf{A}_i\mathbf{h} = 0$$

Direct Linear Transformation (DLT)

- Equations are linear in h

$$\mathrm{A}_i \mathrm{h} = 0$$

- Only 2 out of 3 are linearly independent (indeed, 2 eq/pt)

$$\begin{bmatrix} 0^\top & -w_i'\mathrm{x}_i^\top & y_i'\mathrm{x}_i^\top \\ w_i'\mathrm{x}_i^\top & 0^\top & -x_i'\mathrm{x}_i^\top \\ -y_i'\mathrm{x}_i^\top & x_i'\mathrm{x}_i^\top & 0^\top \end{bmatrix} \begin{pmatrix} \mathrm{h}^1 \\ \mathrm{h}^2 \\ \mathrm{h}^3 \end{pmatrix} = 0$$

$$y_i'\mathrm{A}_i^1 - x_i'\mathrm{A}_i^2 + w_i'\mathrm{A}_i^3 = 0$$

(only drop third row if $w_i \neq 0$)

- Holds for any homogeneous representation, e.g. $(x_i', y_i', 1)$

Direct Linear Transformation (DLT)

- Solving for H

$$\begin{bmatrix} \mathrm{A}_1 \\ \mathrm{A}_2 \\ \mathrm{A}_3 \\ \mathrm{A}_4 \end{bmatrix} \mathrm{A}\mathbf{h} = 0$$

size A is 8x9 or 12x9, but rank 8

Trivial solution is $\mathrm{h}=0_9^T$ is not interesting

1-D null-space yields solution of interest

pick for example the one with $\|\mathrm{h}\| = 1$

Direct Linear Transformation (DLT)

- Over-determined solution

$$\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \mathbf{h} = \mathbf{0}$$

No exact solution because of inexact measurement i.e. "noise"

Find approximate solution

- Additional constraint needed to avoid 0, e.g. $\|\mathbf{h}\| = 1$
- $\mathbf{A}\mathbf{h} = 0$ not possible, so minimize $\|\mathbf{A}\mathbf{h}\|$

DLT algorithm

<u>Objective</u>

Given $n \geq 4$ 2D to 2D point correspondences $\{x_i \leftrightarrow x_i'\}$, determine the 2D homography matrix H such that $x_i' = Hx_i$

<u>Algorithm</u>

(i) For each correspondence $x_i \leftrightarrow x_i'$ compute A_i. Usually only two first rows needed.

(ii) Assemble n 2x9 matrices A_i into a single $2n$x9 matrix A

(iii) Obtain SVD of A. Solution for h is last column of V

(iv) Determine H from h

Inhomogeneous solution

Since h can only be computed up to scale, pick $h_j=1$, e.g. $h_9=1$, and solve for 8-vector $\tilde{h}$

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w_i' & -y_i w_i' & -w_i w_i' & x_i y_i' & y_i y_i' \\ x_i w_i' & y_i w_i' & w_i w_i' & 0 & 0 & 0 & x_i x_i' & y_i x_i' \end{bmatrix} \tilde{h} = \begin{pmatrix} -w_i y_i' \\ w_i x_i' \end{pmatrix}$$

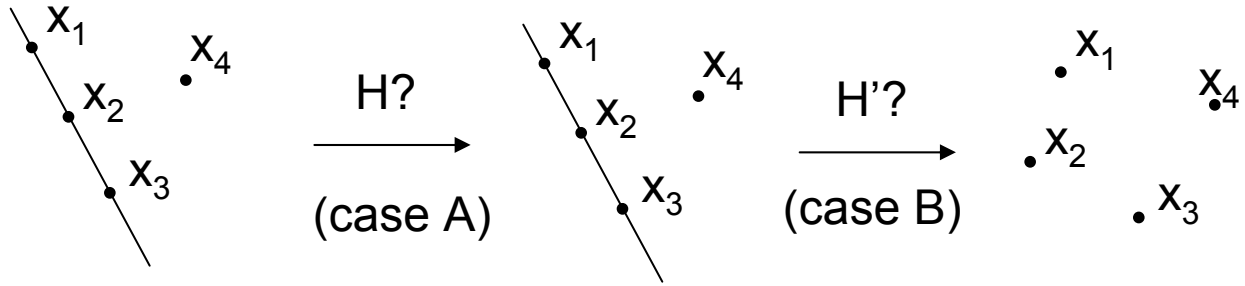
Solve using Gaussian elimination (4 points) or using linear least-squares (more than 4 points)

However, if $h_9=0$ this approach fails
also poor results if h_9 close to zero
Therefore, not recommended

Note $h_9=H_{33}=0$ if origin is mapped to infinity

$$l_\infty^T H x_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} H \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Degenerate configurations



Constraints: $x'_i \times Hx_i = 0$ i=1,2,3,4

Define: $H^* = x'_4 l^T$

Then, $H^* x_i = x'_4 \left(l^T x_i \right) = 0,\ i = 1,2,3$

$H^* x_4 = x'_4 \left(l^T x_4 \right) = k x'_4$

H^* is rank-1 matrix and thus not a homography

If H^* is unique solution, then no homography mapping $x_i \rightarrow x_i'$(case B)
If further solution H exist, then also $\alpha H^* + \beta H$ (case A)
(2-D null-space in stead of 1-D null-space)

Solutions from lines, etc.

2D homographies from 2D lines

$$\mathrm{l}'_i = \mathrm{H}^{\mathsf{T}}\mathrm{l}_i \qquad \mathrm{Ah} = 0$$

Minimum of 4 lines

3D Homographies (15 dof)

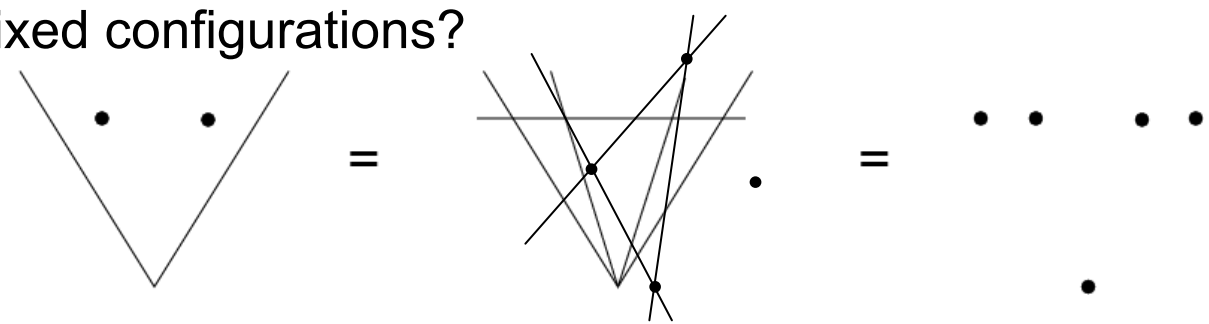
Minimum of 5 points or 5 planes

2D affinities (6 dof)

Minimum of 3 points or lines

Conic provides 5 constraints

Mixed configurations?



Cost functions

- Algebraic distance
- Geometric distance
- Reprojection error

- Comparison
- Geometric interpretation
- Sampson error

Algebraic distance

DLT minimizes $\|\mathrm{Ah}\|$

$e = \mathrm{Ah}$ residual vector

e_i partial vector for each ($x_i \leftrightarrow x_i'$)

algebraic error vector

$$d_{\mathrm{alg}}(\mathrm{x}_i', \mathrm{Hx}_i)^2 = \|e_i\|^2 = \left\| \begin{bmatrix} 0^\mathsf{T} & -w_i'\mathrm{x}_i^\mathsf{T} & -y_i'\mathrm{x}_i^\mathsf{T} \\ -w_i'\mathrm{x}_i^\mathsf{T} & 0^\mathsf{T} & -x_i'\mathrm{x}_i^\mathsf{T} \end{bmatrix} \mathrm{h} \right\|^2$$

algebraic distance

$d_{\mathrm{alg}}(\mathrm{x}_1, \mathrm{x}_2)^2 = a_1^2 + a_2^2$ where $\mathrm{a} = (a_1, a_2, a_3)^\mathrm{T} = \mathrm{x}_1 \times \mathrm{x}_2$

$$\sum_i d_{\mathrm{alg}}(\mathrm{x}_i', \mathrm{Hx}_i)^2 = \sum_i \|e_i\|^2 = \|\mathrm{Ah}\|^2 = \|e\|^2$$

Not geometrically/statistically meaningfull, but given good normalization it works fine and is very fast (use for initialization)

Geometric distance

x measured coordinates

$\hat{\mathrm{x}}$ estimated coordinates

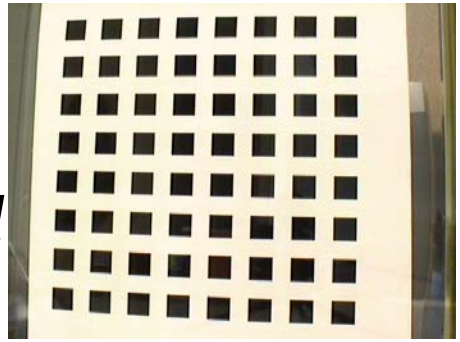
$\overline{\mathrm{x}}$ true coordinates

$d(.,.)$ Euclidean distance (in image)

Error in one image

$$\hat{\mathrm{H}} = \underset{\mathrm{H}}{\operatorname{argmin}} \sum_i d\left(\mathrm{x}'_i, \mathrm{H}\overline{\mathrm{x}}_i\right)^2$$

e.g. calibration pattern



Symmetric transfer error

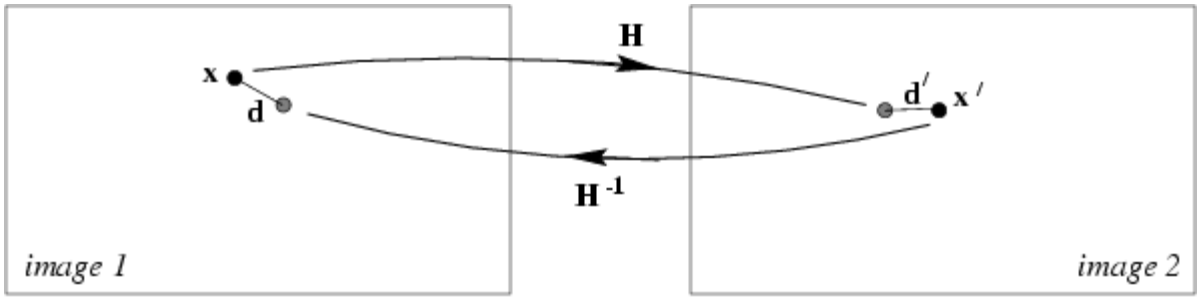
$$\hat{\mathrm{H}} = \underset{\mathrm{H}}{\operatorname{argmin}} \sum_i d\left(\mathrm{x}_i, \mathrm{H}^{-1}\mathrm{x}'_i\right)^2 + d$$

Reprojection error

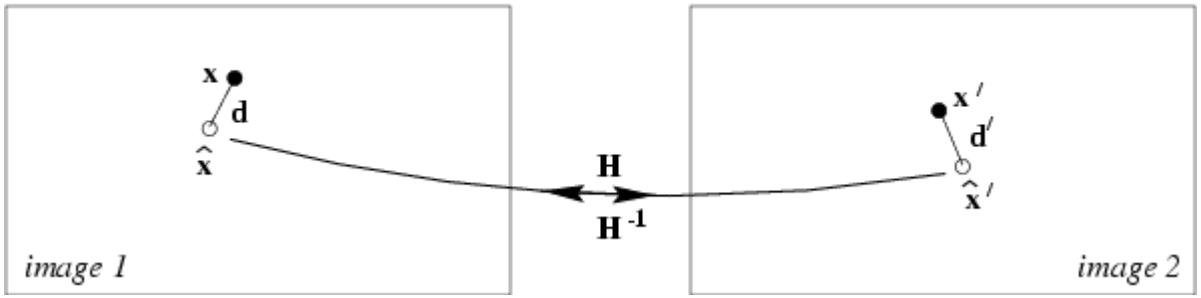
$$\left(\hat{\mathrm{H}}, \hat{\mathrm{x}}_i, \hat{\mathrm{x}}'_i\right) = \underset{\mathrm{H}, \hat{\mathrm{x}}_i, \hat{\mathrm{x}}'_i}{\operatorname{argmin}} \sum_i d\left(\mathrm{x}_i, \hat{\mathrm{x}}_i\right)^2 + d\left(\mathrm{x}'_i, \hat{\mathrm{x}}'_i\right)^2$$

$$\text{subject to } \hat{\mathrm{x}}'_i = \hat{\mathrm{H}}\hat{\mathrm{x}}_i$$

Reprojection error



$$d\left(\mathrm{x}, \mathrm{H}^{-1}\mathrm{x}'\right)^2 + d\left(\mathrm{x}', \mathrm{H}\mathrm{x}\right)^2$$



$$d\left(\mathrm{x}, \hat{\mathrm{x}}\right)^2 + d\left(\mathrm{x}', \hat{\mathrm{x}}'\right)^2$$

Comparison of geometric and algebraic distances

Error in one image

$$\mathrm{x}'_i = \left(x'_i, y'_i, w'_i\right)^{\mathsf{T}} \quad \hat{\mathrm{x}}'_i = \left(\hat{x}'_i, \hat{y}'_i, \hat{w}'_i\right)^{\mathsf{T}} = \mathrm{H}\overline{\mathrm{x}}$$

$$\mathrm{A}_i \mathrm{h} = e_i = \begin{bmatrix} 0^{\mathsf{T}} & -w'_i \mathrm{x}_i^{\mathsf{T}} & y'_i \mathrm{x}_i^{\mathsf{T}} \\ w'_i \mathrm{x}_i^{\mathsf{T}} & 0^{\mathsf{T}} & -x'_i \mathrm{x}_i^{\mathsf{T}} \end{bmatrix} \begin{pmatrix} \mathrm{h}^1 \\ \mathrm{h}^2 \\ \mathrm{h}^3 \end{pmatrix}$$

$$d_{\mathrm{alg}}\left(\mathrm{x}'_{\mathrm{i}}, \hat{\mathrm{x}}'_{\mathrm{i}}\right)^2 = \left(y'_i \hat{w}'_i - w'_i \hat{y}'_i\right)^2 + \left(w'_i \hat{x}'_i - x'_i \hat{w}'_i\right)^2$$

$$d\left(\mathrm{x}'_{\mathrm{i}}, \hat{\mathrm{x}}'_{\mathrm{i}}\right)^2 = \left(\left(y'_i / w'_i - \hat{y}'_i / \hat{w}'_i\right)^2 + \left(\hat{x}'_i / \hat{w}'_i - x'_i / w'_i\right)^2\right)^{1/2}$$

$$= d_{\mathrm{alg}}\left(\mathrm{x}'_{\mathrm{i}}, \hat{\mathrm{x}}'_{\mathrm{i}}\right) / w'_i \hat{w}'_i$$

$w'_i = 1$ typical, but not $\hat{w}'_i = \mathrm{h}_3 \mathrm{x}_{\mathrm{i}}$, except for affinities

For affinities DLT can minimize geometric distance

Possibility for iterative algorithm

Geometric interpretation of reprojection error

Estimating homography~fit surface ν_H to points $X=(x,y,x',y')^T$ in $\mathbb{R}^4$

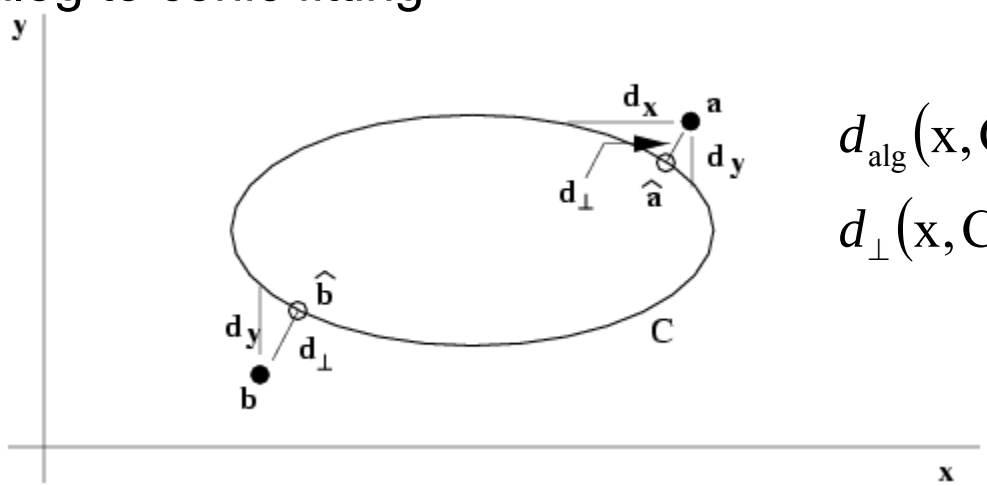
$x'_i \times Hx_i = 0$ represents 2 quadrics in $\mathbb{R}^4$ (quadratic in X)

$$\left\|X_i - \hat{X}_i\right\|^2 = (x_i - \hat{x}_i)^2 + (y_i - \hat{y}_i)^2 + (x'_i - \hat{x}'_i)^2 + (y'_i - \hat{y}'_i)^2$$

$$= d(x_i, \hat{x}_i)^2 + d(x'_i, \hat{x}'_i)^2$$

$$d(x_i, \hat{x}_i)^2 + d(x'_i, \hat{x}'_i)^2 = d_\perp(X_i, \nu_H)^2$$

Analog to conic fitting



$$d_{alg}(x, C)^2 = x^T C x$$

$$d_\perp(x, C)^2$$

Sampson error

between algebraic and geometric error

Vector $\hat{\mathrm{X}}$ that minimizes the geometric error $\left\|\mathrm{X}-\hat{\mathrm{X}}\right\|^2$ is the closest point on the variety $\mathcal{V}_{\mathrm{H}}$ to the measurement X

Sampson error: 1st order approximation of $\hat{\mathrm{X}}$

$$\mathrm{Ah} = C_{\mathrm{H}}(\mathrm{X}) = 0$$

$$C_{\mathrm{H}}(\mathrm{X}+\delta_{\mathrm{X}}) = C_{\mathrm{H}}(\mathrm{X}) + \frac{\partial C_{\mathrm{H}}}{\partial \mathrm{X}}\delta_{\mathrm{X}} \qquad \delta_{\mathrm{X}} = \hat{\mathrm{X}} - \mathrm{X} \qquad C_{\mathrm{H}}(\hat{\mathrm{X}}) = 0$$

$$C_{\mathrm{H}}(\mathrm{X}) + \frac{\partial C_{\mathrm{H}}}{\partial \mathrm{X}}\delta_{\mathrm{X}} = 0 \qquad \mathrm{J}\delta_{\mathrm{X}} = -e \quad \text{with } \mathrm{J} = \frac{\partial C_{\mathrm{H}}}{\partial \mathrm{X}}$$

Find the vector δ_{X} that minimizes $\left\|\delta_{\mathrm{X}}\right\|$ subject to $\mathrm{J}\delta_{\mathrm{X}} = -e$

Find the vector δ_X that minimizes $\|\delta_X\|$ subject to $J\delta_X = -e$

Use Lagrange multipliers:

minimize $\quad \delta_X{}^T\delta_X - 2\lambda\left(J\delta_X + e\right) = 0$

derivatives $\quad 2\delta_X - 2\lambda^T J = 0^T \quad \Rightarrow \delta_X = J^T\lambda$

$$2\left(J\delta_X + e\right) = 0 \quad \Rightarrow JJ^T\lambda + e = 0$$

$$\Rightarrow \lambda = -\left(JJ^T\right)^{-1} e$$

$$\Rightarrow \delta_X = -J^T\left(JJ^T\right)^{-1} e$$

$$\hat{X} = X + \delta_X \qquad \|\delta_X\|^2 = \delta_X^T\delta_X = e^T\left(JJ^T\right)^{-1} e$$

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between algebraic and geometric error

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$$\mathrm{Ah} = C_{\mathrm{H}}(\mathrm{X}) = 0$$

$$C_{\mathrm{H}}(\mathrm{X}+\delta_{\mathrm{X}}) = C_{\mathrm{H}}(\mathrm{X}) + \frac{\partial C_{\mathrm{H}}}{\partial \mathrm{X}}\delta_{\mathrm{X}} \qquad \delta_{\mathrm{X}} = \hat{\mathrm{X}} - \mathrm{X} \qquad C_{\mathrm{H}}(\hat{\mathrm{X}}) = 0$$

$$C_{\mathrm{H}}(\mathrm{X}) + \frac{\partial C_{\mathrm{H}}}{\partial \mathrm{X}}\delta_{\mathrm{X}} = 0 \qquad \mathrm{J}\delta_{\mathrm{X}} = -e$$

Find the vector δ_{X} that minimizes $\left\|\delta_{\mathrm{X}}\right\|$ subject to $\mathrm{J}\delta_{\mathrm{X}} = -e$

$$\left\|\delta_{\mathrm{X}}\right\|^2 = \delta_{\mathrm{X}}^{\mathrm{T}}\delta_{\mathrm{X}} = e^{\mathrm{T}}\left(\mathrm{JJ}^{\mathrm{T}}\right)^{-1}e \qquad \text{(Sampson error)}$$

Sampson approximation

$$\left\|\delta_{\mathrm{X}}\right\|^2 = e^{\mathrm{T}}\left(\mathrm{J}\mathrm{J}^{\mathrm{T}}\right)^{-1} e$$

A few points

(i) For a 2D homography $\mathrm{X}=(\mathrm{x},\mathrm{y},\mathrm{x}',\mathrm{y}')$

(ii) $e = C_{\mathrm{H}}(\mathrm{X})$ is the algebraic error vector

(iii) $\mathrm{J} = \dfrac{\partial C_{\mathrm{H}}}{\partial \mathrm{X}}$ is a 2x4 matrix,
e.g. $J_{11} = \partial\left(-w_i'\mathrm{x}_i{}^{\mathrm{T}}\mathrm{h}^2 + y_i'\mathrm{x}_i{}^{\mathrm{T}}\mathrm{h}^3\right)/\partial x = -w_i'h_{21} + y_i'h_{31}$

(iv) Similar to algebraic error $\|e\|^2 = e^{\mathrm{T}}e$
in fact, same as Mahalanobis distance $\|e\|^2_{\mathrm{J}\mathrm{J}^{\mathrm{T}}}$

(v) Sampson error independent of linear reparametrization
(cancels out in between e and J)

(vi) Must be summed for all points $\sum e^{\mathrm{T}}\left(\mathrm{J}\mathrm{J}^{\mathrm{T}}\right)^{-1} e$

(vii) Close to geometric error, but much fewer unknowns

Statistical cost function and Maximum Likelihood Estimation

- Optimal cost function related to noise model
- Assume zero-mean isotropic Gaussian noise (assume outliers removed)

$$\Pr(\mathrm{x}) = \frac{1}{2\pi\sigma^2} e^{-d(\mathrm{x},\bar{\mathrm{x}})^2/(2\sigma^2)}$$

Error in one image

$$\Pr(\{\mathrm{x}'_i\} \mid \mathrm{H}) = \prod_i \frac{1}{2\pi\sigma^2} e^{-d(\mathrm{x}'_i, \mathrm{H}\bar{\mathrm{x}}_i)^2/(2\sigma^2)}$$

$$\log \Pr(\{\mathrm{x}'_i\} \mid \mathrm{H}) = -\frac{1}{2\sigma^2} \sum d(\mathrm{x}'_i, \mathrm{H}\bar{\mathrm{x}}_i)^2 + \text{constant}$$

Maximum Likelihood Estimate

$$\sum d(\mathrm{x}'_i, \mathrm{H}\bar{\mathrm{x}}_i)^2$$

Statistical cost function and Maximum Likelihood Estimation

- Optimal cost function related to noise model
- Assume zero-mean isotropic Gaussian noise (assume outliers removed)

$$\Pr(\mathrm{x}) = \frac{1}{2\pi\sigma^2} e^{-d(\mathrm{x},\bar{\mathrm{x}})^2/(2\sigma^2)}$$

Error in both images

$$\Pr(\{\mathrm{x}'_i\} \mid \mathrm{H}) = \prod_i \frac{1}{2\pi\sigma^2} e^{-\left(d(\mathrm{x}_i,\bar{\mathrm{x}}_i)^2 + d(\mathrm{x}'_i,\mathrm{H}\bar{\mathrm{x}}_i)^2\right)/(2\sigma^2)}$$

Maximum Likelihood Estimate

$$\sum d(\mathrm{x}_i,\hat{\mathrm{x}}_i)^2 + d(\mathrm{x}'_i,\hat{\mathrm{x}}'_i)^2$$

Mahalanobis distance

- General Gaussian case

  Measurement X with covariance matrix Σ

$$\left\|\mathrm{X}-\overline{\mathrm{X}}\right\|_{\Sigma}^{2}=\left(\mathrm{X}-\overline{\mathrm{X}}\right)^{\mathrm{T}}\Sigma^{-1}\left(\mathrm{X}-\overline{\mathrm{X}}\right)$$

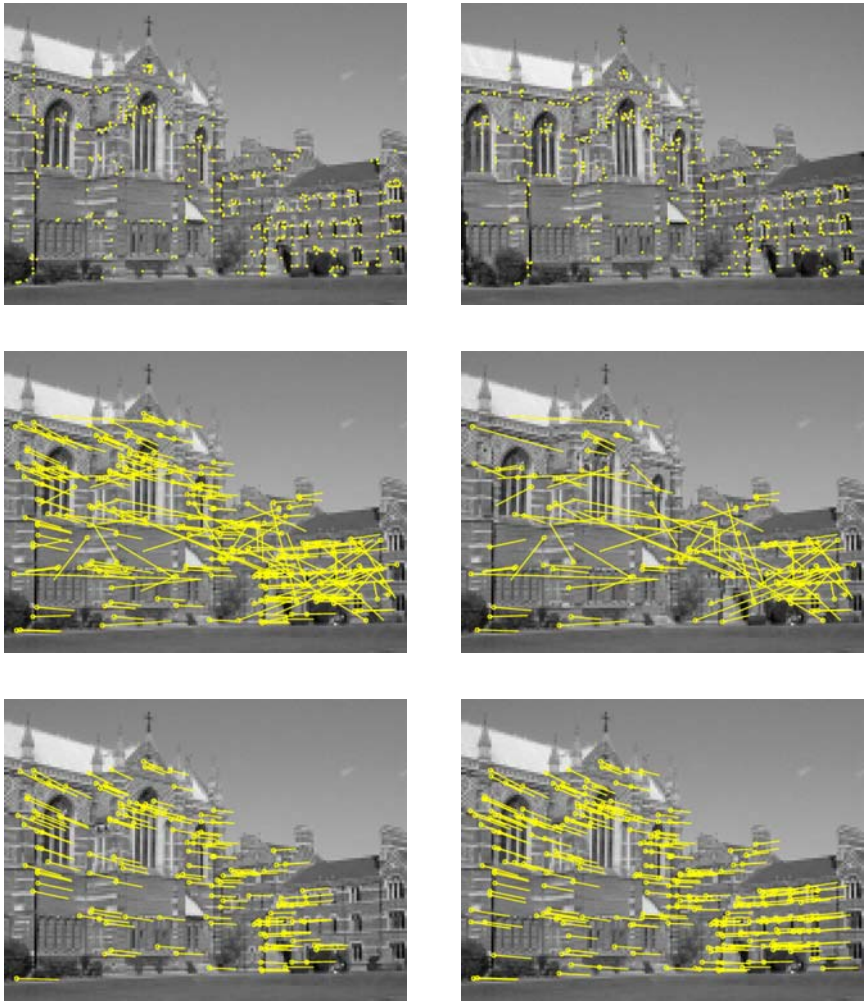
Error in two images (independent)

$$\left\|\mathrm{X}-\overline{\mathrm{X}}\right\|_{\Sigma}^{2}+\left\|\mathrm{X}'-\overline{\mathrm{X}}'\right\|_{\Sigma'}^{2}$$

Varying covariances

$$\sum_{i}\left\|\mathrm{X}_{i}-\overline{\mathrm{X}}_{i}\right\|_{\Sigma_{i}}^{2}+\left\|\mathrm{X}_{i}'-\overline{\mathrm{X}}_{i}'\right\|_{\Sigma_{i}'}^{2}$$

Next class:
Parameter estimation (continued)



Transformation invariance and normalization

Iterative minimization

Robust estimation

Upcoming assignment

- Take two or more photographs taken from a single viewpoint
- Compute panorama
- Use different measures DLT, MLE

- Use Matlab
- Due Feb. 13