EECS 290S: Network Information Flow **Fall 2008**

Lecture 23 — November 18

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23.1 General Layered Network

In this lecture, we analyze the capacity of the network shown in Figure 23.1, when sending data from the source S to the destination D. In this lecture, we will study the network under the Gaussian model, where Gaussian noise is added to each received signal. For example, the signal Y_{B_2} received by node B_2 is a composition of input signals and Gaussian noise, as shown in Figure 23.2. In general, we let Y_i represent the signal received by node i, and X_i represent the signal sent by node i.

We start by proving an upper bound on the achievable rate, which is not tight but gives a rough upper bound on the achievable rate. As usual, we can apply the same cut set bound as we have done previously, to bound the maximum achievable rate R:

$$\begin{aligned} R &\leq \max_{p(\{X_i\})} \min_{\Omega} I(Y_{\Omega^c}; X_\Omega | X_{\Omega^c}) && (23.1) \\ &\leq \min_{\Omega} \max_{p(\{X_i\})} I(Y_{\Omega^c}; X_\Omega | X_{\Omega^c}) && (23.2) \end{aligned}$$

In the statement above, the maximum is taken over all distributions $p(\{X_i\})$, which obey the power constraint $\mathbb{E}[|X_i|^2] \leq P$, where X_i is the codeword transmitted by node i and P is a given power limit. The minimum is taken over all cuts $\Omega \subset V$ such that $s \in \Omega$ and $t \notin \Omega$. The first line follows from basic information theory concepts, and the second line follows since switching the order of the max and min terms only increases the value of the expression. As we will see in the this and the next lecture, this upper bound is almost

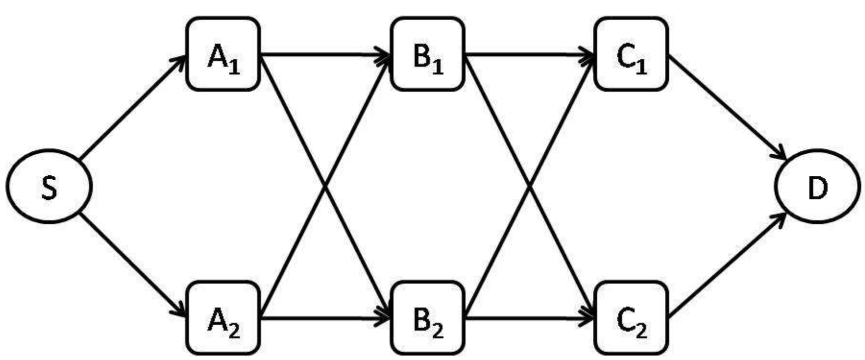

Figure 23.1. A layered network in which we want to send data from the source S to the destination D

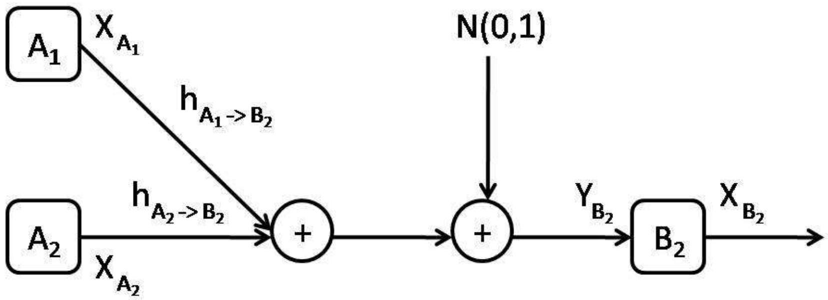

Figure 23.2. Illustration of how the signal Y_{B_2} to B_2 is composed. We assume node A_1 sends the message X_{A_1} with gain $h_{A_1 \to B_2}$ and node A_2 sends X_{A_2} with gain $h_{A_2 \to B_2}$, and one unit of Gaussian noise $N(0,1)$ is added.

tight, as there are coding schemes which achieve a rate, which is within a constant times the number of nodes in the layered network, of this upper bound.

To derive a coding scheme for our leveled network, one idea could be to simply have each node amplify the messages which it receives. However, this clearly does not do well, as the noise gets amplified and can easily drown out the real signal/message. Another idea would be to transit in blocks of T symbols at a time, as follows:

1. Have the source node transmit a random codeword based on the message ω
2. Have each relay node i:
   (a) Quantize Y_i to Y_i^Q with quantization points spaced by 1
   (b) Transmit a random codeword based on the quantization $Y_i^Q(0), \ldots, Y_i^Q(T)$
3. Destination D decodes a message $\hat{\omega}$

In the scheme above, we draw all random codewords i.i.d. from the distribution $N(0, P)$. If we employ the above scheme, we experience three forms of loss, which causes a gap between the achievable rate and the cut set bound. We categorize these losses into three categories:

(i) Beamforming Gain Loss (Waterfilling)

(ii) Discretization Loss

(iii) "Header Accumulation" Loss

For the loss of type (ii), only a constant number of bits is lost overall, but the losses of type (i) and (iii) may cause losses which are proportional to the number of levels or nodes in the network.

23.1.1 Beamforming Gain Loss

To estimate the loss that might arise from not being able to beamform or waterfill, note that if we look at a particular cut Ω, the capacity of the cut is $C_\Omega = \max_{p(\{X_i\})} I(Y_{\Omega^c}; X_\Omega | X_{\Omega^c})$. Note that the term $I(Y_{\Omega^c}; X_\Omega | X_{\Omega^c})$ can be upper bounded, as follows:

$$\begin{aligned} I(Y_{\Omega^c}; X_\Omega | X_{\Omega^c}) &= h(Y_{\Omega^c} | X_{\Omega^c}) - h(Y_{\Omega^c} | X_\Omega, X_{\Omega^c}) && (23.3)\\ &\leq h(Y'_{\Omega^c}) - h(Y'_{\Omega^c} | X_\Omega) && (23.4)\\ &\leq I(Y'_{\Omega^c}; X_\Omega) && (23.5) \end{aligned}$$

where Y'_{Ω^c} are the outputs that arise if we set $X_i = 0$ for all $i \in \Omega^c$.

Point to Point MIMO Channel

To calculate the loss associated with not being able to utilize beamforming or waterfilling, let us look at a MIMO (multiple input, multiple output) channel, and let us analyze the gap in rates when beamforming/waterfilling is allowed and when it is not allowed. In particular, let us suppose our input is $|\Omega|$-dimensional and the output is $|\Omega^c|$-dimensional, and η_i is the noise associated with node i's transmission, for nodes $i = 1, \ldots, n = |\Omega|$. In one setting, we fix all n nodes to have power exactly P, while in another setting we allow the n nodes to arbitrarily divide nP units of power, so that each node i transits with some power μ_i such that $\sum_{i=1}^n \mu_i = nP$. If C_Ω is the achievable rate when the power can be arbitrarily divided, and $C_{\text{Equal Power}}$ is the rate achievable when each node has an equal amount of power P, then the gap between the two rates C_Ω and $C_{\text{Equal Power}}$, can be bounded as follows:

$$\begin{aligned} C_\Omega - C_{\text{Equal Power}} &\leq \log(\prod_{i=1}^n (1 + \frac{\mu_i}{\eta_i})) - \log(\prod_{i=1}^n (1 + \frac{P}{\eta_i})) && (23.6)\\ &\leq \log(\prod_{i=1}^n ((1 + \frac{\mu_i}{\eta_i})/(1 + \frac{P}{\eta_i}))) && (23.7)\\ &< \log(\prod_{i=1}^n (1 + \frac{\mu_i}{P})) && (23.8)\\ &< \sum_{i=1}^n (1 + \mu_i/P) = n && (23.9) \end{aligned}$$

The first line follows from basic information theory, and the second line follows by combining terms. To go from the second line to the third line, note that for $a \geq 0$ and $b > 0$, $\frac{1+a}{1+b} \leq 1 + \frac{a}{b}$, since $1 + a < 1 + b + \frac{a}{b} + a$. To get from the third line to the last line, note that $\log(1 + x) < (1 + x)$, for $x > 0$.

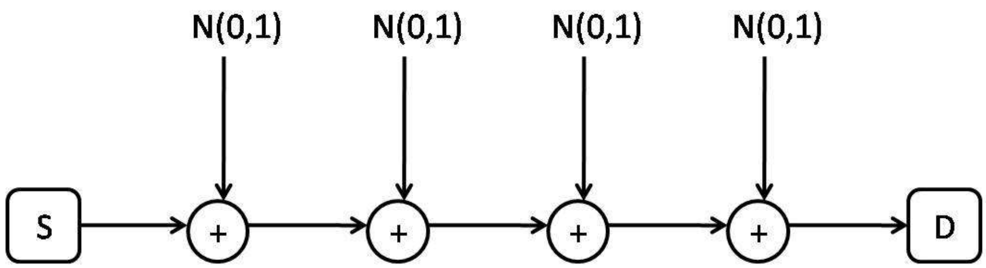

Figure 23.3. If the relay nodes simply forward the messages received, then destination node receives a signal with $N(0,n)$ noise added, and only $\log(1+P/n)$ rate can be achieved.

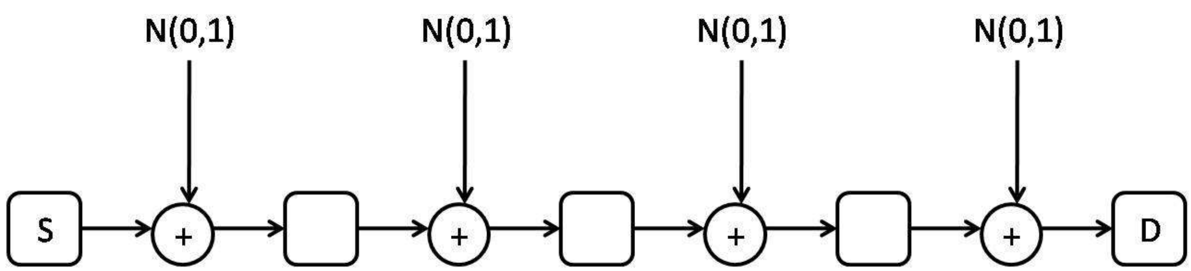

Figure 23.4. If the relay nodes decode the messages and forward the decoded messages, then a much better rate of $\log(1+P)$ can be achieved.

23.2 Linear Relay Network

In a linear relay network we can compare the rates achievable by the two strategies, decode and forward, and simply forward. If we simply forward the received signal (ignoring power constraints), the situation looks like Figure 23.3 and the achievable rate is $\log(1+P/n)$, where n is the number of nodes in the relay. If the relay nodes decode at each step and then forward, the situation looks like Figure 23.4, and we can achieve a better rate of $\log(1+P)$. The difference in achievable rates can be bounded by $O(\log n)$, as the loss is $\log(1+P) - \log(1+P/n) = \log(\frac{1+P}{1+P/n}) < \log(1+n)$, using the inequality derived in the last section.