Remark 2 Even if we allow an arbitrarily small probability of decoding error in the usual Shannon sense, by modifying our proof by means of a standard application of Fano's inequality, it can be shown that it is still necessary for ω to satisfy (11.51). The details are omitted here.

11.5 ACHIEVABILITY OF THE MAX-FLOW BOUND

The max-flow bound has been proved in the last section. The achievability of this bound is stated in the following theorem. This theorem will be proved after the necessary preliminaries are presented.

THEOREM 11.4 *For a graph G with rate constraints* $\mathbf{R}$, *if*

$$\omega \leq \min_l \mathrm{maxflow}(s, t_l), \tag{11.54}$$

then ω is achievable.

A *directed path* in a graph G is a finite non-null sequence of nodes

$$v_1, v_2, \cdots, v_{m-1}, v_m \tag{11.55}$$

such that $(v_i, v_{i+1}) \in E$ for all $i = 1, 2, \cdots, m-1$. The edges (v_i, v_{i+1}), $i = 1, 2, \cdots, m-1$, are referred to as the edges on the directed path. Such a sequence is called a directed path from v_1 to v_m. If $v_1 = v_m$, then it is called a *directed cycle*. If there exists a directed cycle in G, then G is *cyclic*, otherwise G is *acyclic*.

In Section 11.5.1, we will first prove Theorem 11.4 for the special case when the network is acyclic. Acyclic networks are easier to handle because the nodes in the network can be ordered in a way which allows coding at the nodes to be done in a sequential and consistent manner. The following proposition describes such an order, and the proof shows how it can be obtained. In Section 11.5.2, Theorem 11.4 will be proved in full generality.

PROPOSITION 11.5 *If $G = (V, E)$ is a finite acyclic graph, then it is possible to order the nodes of G in a sequence such that if there is an edge from node i to node j, then node i is before node j in the sequence.*

Proof We partition V into subsets $V_1, V_2, \cdots$, such that node i is in V_k if and only if the length of a longest directed path ending at node i is equal to k. We claim that if node i is in $V_{k'}$ and node j is in V_k such that $k \leq k'$, then there exists no directed path from node i to node j. This is proved by contradiction as follows. Assume that there exists a directed path from node i to node j. Since there exists a directed path ending at node i of length k', there exists a directed path ending at node j containing node i whose length is at least $k'+1$.

Since node j is in V_k, the length of a longest directed path ending at node j is equal to k. Then

$$k' + 1 \leq k. \tag{11.56}$$

However, this is a contradiction because

$$k' + 1 > k' \geq k. \tag{11.57}$$

Therefore, we conclude that there exists a directed path from a node in $V_{k'}$ to a node in V_k, then $k' < k$.

Hence, by listing the nodes of G in a sequence such that the nodes in V_k appear before the nodes in $V_{k'}$ if $k < k'$, where the order of the nodes within each V_k is arbitrary, we obtain an order of the nodes of G with the desired property. □

EXAMPLE 11.6 *Consider ordering the nodes in the acyclic graph in Figure 11.3 by the sequence*

$$s, 2, 1, 3, 4, t_2, t_1. \tag{11.58}$$

It is easy to check that in this sequence, if there is a directed path from node i to node j, then node i appears before node j.

11.5.1 ACYCLIC NETWORKS

In this section, we prove Theorem 11.4 for the special case when the graph G is acyclic. Let the vertices in G be labeled by $0, 1, \cdots, |V| - 1$ in the following way. The source s has the label 0. The other vertices are labeled in a way such that for $1 \leq j \leq |V| - 1$, $(i, j) \in E$ implies $i < j$. Such a labeling is possible by Proposition 11.5. We regard $s, t_1, \cdots, t_L$ as aliases of the corresponding vertices.

We will consider an $(n, (\eta_{ij} : (i,j) \in E), \tau)$ *β-code* on the graph G defined by

1) for all $(s, j) \in E$, an encoding function

$$f_{sj} : \mathcal{X} \to \{1, 2, \cdots, \eta_{sj}\}, \tag{11.59}$$

where

$$\mathcal{X} = \{1, 2, \cdots, \lceil 2^{n\tau} \rceil\}; \tag{11.60}$$

2) for all $(i, j) \in E$ such that $i \neq s$, an encoding function

$$f_{ij} : \prod_{i' : (i', i) \in E} \{1, 2, \cdots, \eta_{i'i}\} \to \{1, 2, \cdots, \eta_{ij}\} \tag{11.61}$$

(if $\{i' : (i', i) \in E\}$ is empty, we adopt the convention that f_{ij} is an arbitrary constant taken from $\{1, 2, \cdots, \eta_{ij}\}$);

3) for all $l = 1, 2, \cdots, L$, a decoding function

$$g_l : \prod_{i:(i,t_l) \in E} \{1, 2, \cdots, \eta_{it_l}\} \to \mathcal{X} \tag{11.62}$$

such that

$$\tilde{g}_l(x) = x \tag{11.63}$$

for all $x \in \mathcal{X}$. (Recall that $\tilde{g}_l(x)$ denotes the value of g_l as a function of x.)

In the above, f_{ij} is the encoding function for edge (i, j), and g_l is the decoding function for sink node t_l. In a coding session, f_{ij} is applied before $f_{i'j'}$ if $i < i'$, and f_{ij} is applied before $f_{ij'}$ if $j < j'$. This defines the order in which the encoding functions are applied. Since $i' < i$ if $(i', i) \in E$, a node does not encode until all the necessary information is received on the input channels. A β-code is a special case of an α-code defined in Section 11.3.

Assume that ω satisfies (11.54) with respect to rate constraints $\mathbf{R}$. It suffices to show that for any $\epsilon > 0$, there exists for sufficiently large n an $(n, (\eta_{ij} : (i, j) \in E), \omega - \epsilon)$ β-code on G such that

$$n^{-1} \log_2 \eta_{ij} \le R_{ij} + \epsilon \tag{11.64}$$

for all $(i, j) \in E$. Instead, we will show the existence of an $(n, (\eta_{ij} : (i, j) \in E), \omega)$ β-code satisfying the same set of conditions. This will be done by constructing a random code. In constructing this code, we temporarily replace $\mathcal{X}$ by

$$\mathcal{X}' = \{1, 2, \cdots, \lceil C 2^{n\omega} \rceil\}, \tag{11.65}$$

where C is any constant greater than 1. Thus the domain of f_{sj} is expanded from $\mathcal{X}$ to $\mathcal{X}'$ for all $(s, j) \in E$.

We now construct the encoding functions as follows. For all $j \in V$ such that $(s, j) \in E$, for all $x \in \mathcal{X}'$, let $f_{sj}(x)$ be a value selected independently from the set $\{1, 2, \cdots, \eta_{sj}\}$ according to the uniform distribution. For all $(i, j) \in E, i \neq s$, and for all

$$z \in \prod_{i':(i',i) \in E} \{1, 2, \cdots, \eta_{i'i}\}, \tag{11.66}$$

let $f_{ij}(z)$ be a value selected independently from the set $\{1, 2, \cdots, \eta_{ij}\}$ according to the uniform distribution.

Let

$$z_s(x) = x, \tag{11.67}$$

and for $j \in V, j \neq s$, let

$$z_j(x) = (\tilde{f}_{ij}(x), (i, j) \in E), \tag{11.68}$$

where $x \in \mathcal{X}'$ and $\tilde{f}_{ij}(x)$ denotes the value of f_{ij} as a function of x. $z_j(x)$ is all the information received by node j during the coding session when the message is x. For distinct $x, x' \in \mathcal{X}$, x and x' are indistinguishable at sink t_l if and only if $z_{t_l}(x) = z_{t_l}(x')$. For all $x \in \mathcal{X}$, define

$$F(x) = \begin{cases} 1 & \text{if for some } l = 1, 2, \cdots, L, \text{ there exists } x' \in \mathcal{X}, \\ & x' \neq x, \text{ such that } z_{t_l}(x) = z_{t_l}(x'), \\ 0 & \text{otherwise.} \end{cases} \tag{11.69}$$

$F(x)$ is equal to 1 if and only if x cannot be uniquely determined at at least one of the sink nodes. Now fix $x \in \mathcal{X}$ and $1 \leq l \leq L$. Consider any $x' \in \mathcal{X}$ not equal to x and define the sets

$$U_0 = \{i \in V : z_i(x) \neq z_i(x')\} \tag{11.70}$$

and

$$U_1 = \{i \in V : z_i(x) = z_i(x')\}. \tag{11.71}$$

U_0 is the set of nodes at which the two messages x and x' are distinguishable, and U_1 is the set of nodes at which x and x' are indistinguishable. Obviously, $s \in U_0$.

Now suppose $z_{t_l}(x) = z_{t_l}(x')$. Then $U_0 = U$ for some $U \subset V$, where $s \in U$ and $t_l \notin U$, i.e., U is a cut between node s and node t_l. For any $(i, j) \in E$,

$$\Pr\{\tilde{f}_{ij}(x) = \tilde{f}_{ij}(x') | z_i(x) \neq z_i(x')\} = \eta_{ij}^{-1}. \tag{11.72}$$

Therefore,

$$\begin{aligned} & \Pr\{U_0 = U\} \\ &= \Pr\{U_0 = U, U_0 \supset U\} && (11.73) \\ &= \Pr\{U_0 = U | U_0 \supset U\} \Pr\{U_0 \supset U\} && (11.74) \\ &\leq \Pr\{U_0 = U | U_0 \supset U\} && (11.75) \\ &= \prod_{(i,j) \in E_U} \Pr\{\tilde{f}_{ij}(x) = \tilde{f}_{ij}(x') | z_i(x) \neq z_i(x')\} && (11.76) \\ &= \prod_{(i,j) \in E_U} \eta_{ij}^{-1}, && (11.77) \end{aligned}$$

where

$$E_U = \{(i, j) \in E : i \in U, j \notin U\} \tag{11.78}$$

is the set of all the edges across the cut U as previously defined.

Let ϵ be any fixed positive real number. For all $(i, j) \in E$, take η_{ij} such that

$$R_{ij} + \zeta \leq n^{-1} \log \eta_{ij} \leq R_{ij} + \epsilon \tag{11.79}$$

for some $0 < \zeta < \epsilon$. Then

$$\begin{aligned}
\Pr\{U_0 = U\} &\le \prod_{(i,j)\in E_U} \eta_{ij}^{-1} && (11.80)\\
&\le \prod_{(i,j)\in E_U} 2^{-n(R_{ij}+\zeta)} && (11.81)\\
&= 2^{-n\left(|E_U|\zeta + \sum_{(i,j)\in E_U} R_{ij}\right)} && (11.82)\\
&\stackrel{a)}{\le} 2^{-n\left(\zeta + \sum_{(i,j)\in E_U} R_{ij}\right)} && (11.83)\\
&\stackrel{b)}{\le} 2^{-n(\zeta + \mathrm{maxflow}(s,t_l))} && (11.84)\\
&\stackrel{c)}{\le} 2^{-n(\omega+\zeta)}, && (11.85)
\end{aligned}$$

where

a) follows because $|E_U| \ge 1$;

b) follows because

$$\sum_{(i,j)\in E_U} R_{ij} \ge \min_{U'} \sum_{(i,j)\in E_U} R_{ij} = \mathrm{maxflow}(s, t_l), \qquad (11.86)$$

where U' is a cut between node s and node t_l, by the max-flow min-cut theorem;

c) follows from (11.54).

Note that this upper bound does not depend on U. Since U is some subset of V and V has $2^{|V|}$ subsets,

$$\begin{aligned}
&\Pr\{z_{t_l}(x) = z_{t_l}(x')\}\\
&= \Pr\{U_0 = U \text{ for some cut } U \text{ between node } s \text{ and node } t_l\} && (11.87)\\
&\le 2^{|V|} 2^{-n(\omega+\zeta)} && (11.88)
\end{aligned}$$

by the union bound. Further,

$$\begin{aligned}
&\Pr\{z_{t_l}(x) = z_{t_l}(x') \text{ for some } x' \in \mathcal{X}', x' \ne x\}\\
&\le (|\mathcal{X}'| - 1) 2^{|V|} 2^{-n(\omega+\zeta)} && (11.89)\\
&< C 2^{n\omega} 2^{|V|} 2^{-n(\omega+\zeta)} && (11.90)\\
&= C 2^{|V|} 2^{-n\zeta}, && (11.91)
\end{aligned}$$

where (11.89) follows from the union bound and (11.90) follows from (11.65). Therefore,

$$E[F(x)]$$

$$
\begin{aligned}
&= \Pr\{F(x) = 1\} && (11.92)\\
&= \Pr\left\{\bigcup_{l=1}^{L}\{z_{t_l}(x) = z_{t_l}(x') \text{ for some } x' \in \mathcal{X}', x' \neq x\ \}\right\} && (11.93)\\
&< LC2^{|V|}2^{-n\zeta} && (11.94)\\
&= \delta(n,\zeta) && (11.95)
\end{aligned}
$$

by the union bound, where

$$\delta(n,\zeta) = LC2^{|V|}2^{-n\zeta}. \tag{11.96}$$

Now the total number of messages which can be uniquely determined at all the sink nodes is equal to

$$\sum_{x\in\mathcal{X}'}(1 - F(x)). \tag{11.97}$$

By taking expectation for the random code we have constructed, we have

$$
\begin{aligned}
E\sum_{x\in\mathcal{X}'}(1 - F(x)) &= \sum_{x\in\mathcal{X}'}(1 - E[F(x)]) && (11.98)\\
&> \sum_{x\in\mathcal{X}'}(1 - \delta(n,\zeta)) && (11.99)\\
&\geq (1 - \delta(n,\zeta))C2^{n\omega}, && (11.100)
\end{aligned}
$$

where (11.99) follows from (11.95), and the last step follows from (11.65). Hence, there exists a deterministic code for which the number of messages which can be uniquely determined at all the sink nodes is at least

$$(1 - \delta(n,\zeta))C2^{n\omega}, \tag{11.101}$$

which is greater than $2^{n\omega}$ for n sufficiently large since $\delta(n,\zeta) \to 0$ as $n \to \infty$. Let $\mathcal{X}$ to be any set of $\lceil 2^{n\omega}\rceil$ such messages in $\mathcal{X}'$. For $l = 1, 2, \cdots, L$ and

$$z \in \prod_{i':(i',t_l)\in E}\{1, 2, \cdots, \eta_{i't_l}\}, \tag{11.102}$$

upon defining

$$g_l(z) = x \tag{11.103}$$

where $x \in \mathcal{X}$ such that

$$z = z_{t_l}(x), \tag{11.104}$$

we have obtained a desired $(n, (\eta_{ij} : (i,j) \in E), \omega)$ β-code. Hence, Theorem 11.4 is proved for the special case when G is acyclic.