# Outline

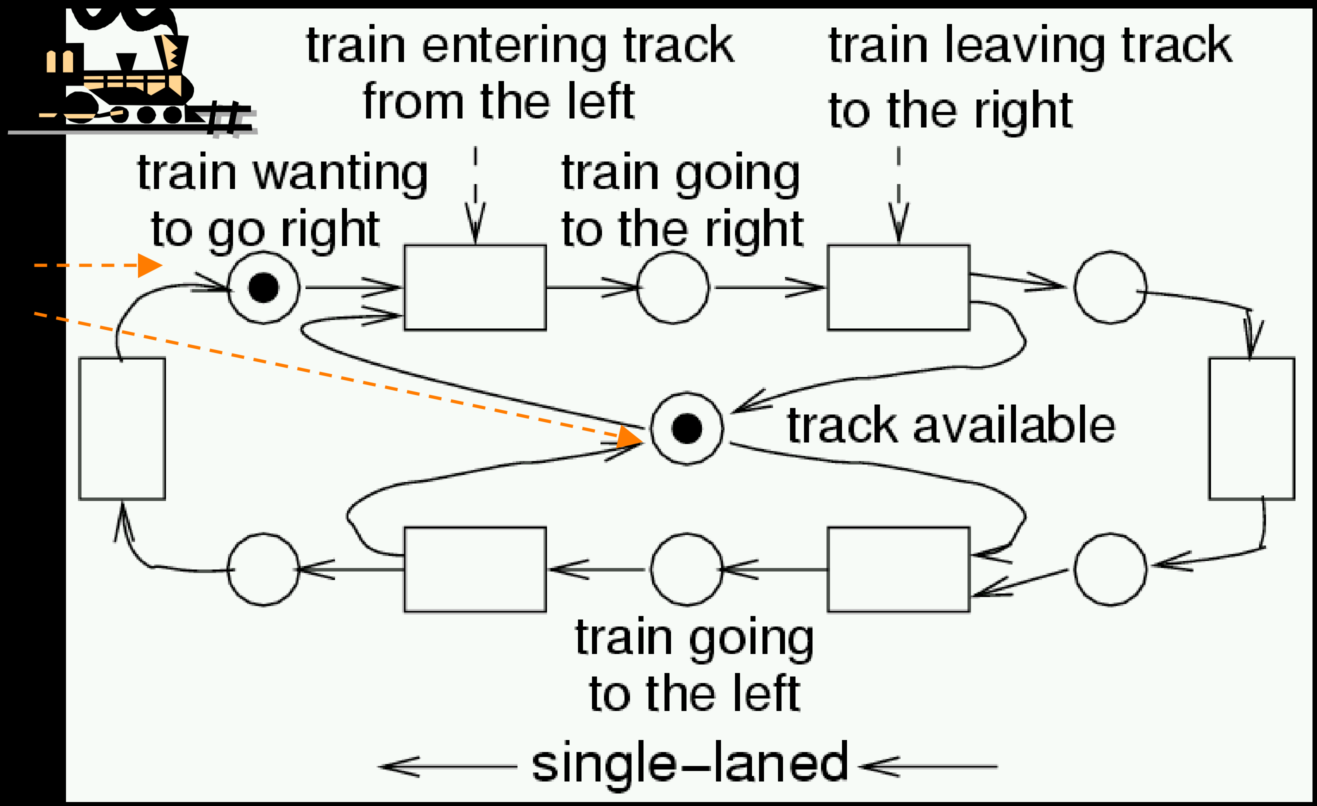
- **Petri nets**
  - **Introduction**
  - **Examples**
  - **Properties**
  - **Analysis techniques**

# Petri Nets (PNs)

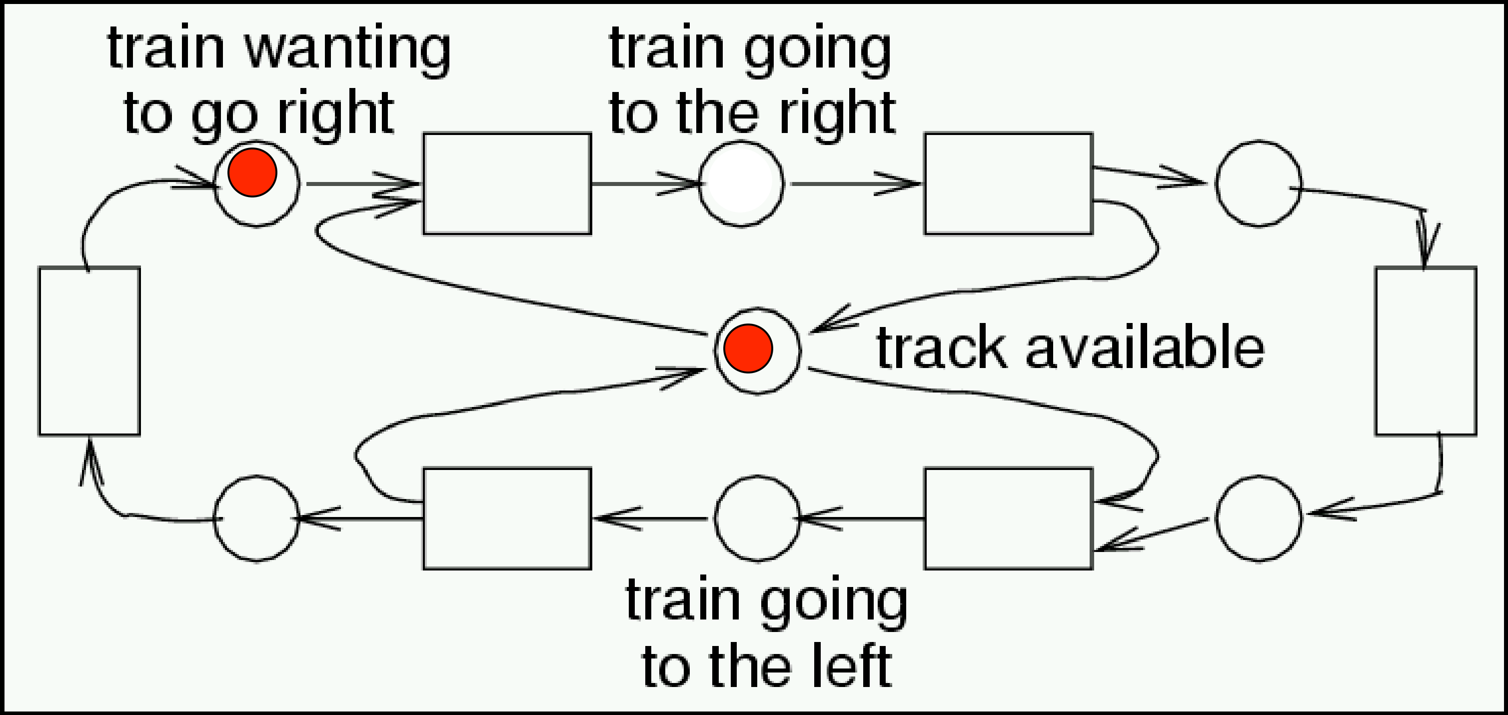
- Model introduced by C.A. Petri in 1962
  - Ph.D. Thesis: “Communication with Automata”
- Applications: distributed computing, manufacturing, control, communication networks, transportation…
- PNs describe explicitly and graphically:
  - sequencing/causality
  - conflict/non-deterministic choice
  - concurrency
- Basic PN model
  - Asynchronous model (partial ordering)
  - Main drawback: no hierarchy

# Example:
# Synchronization at single track rail segment

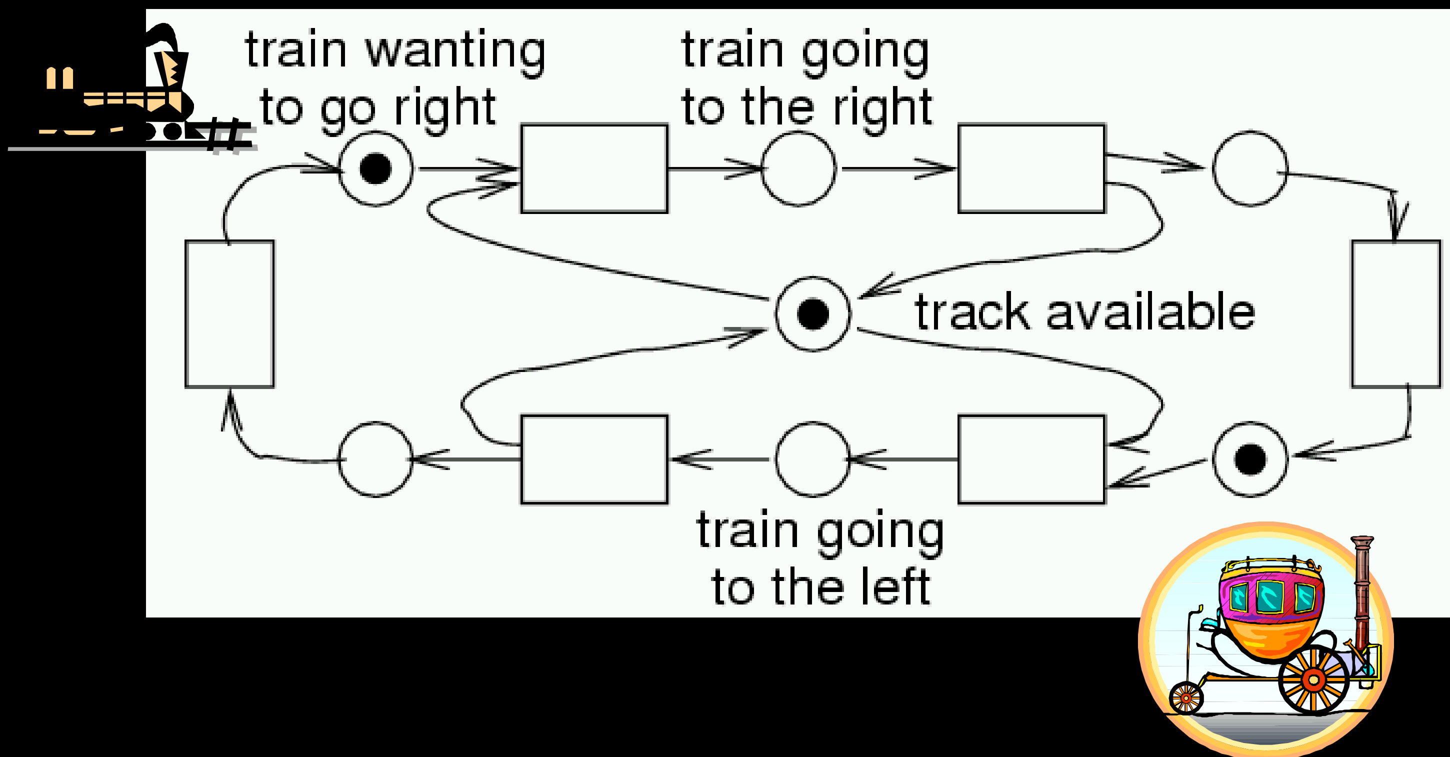
- „Preconditions“



# Playing the „token game“

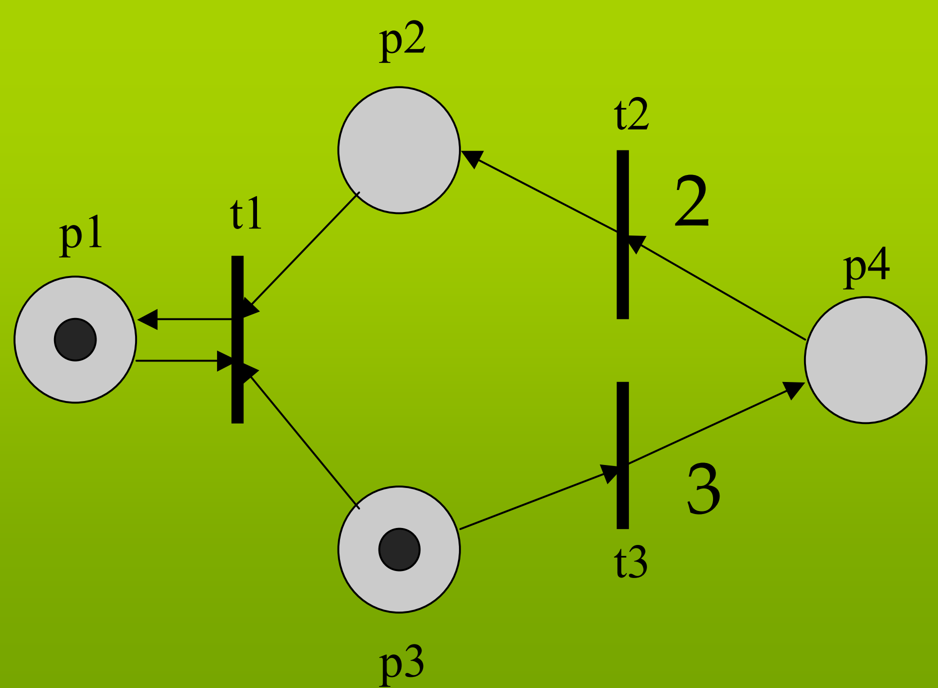


# Conflict for resource „track“



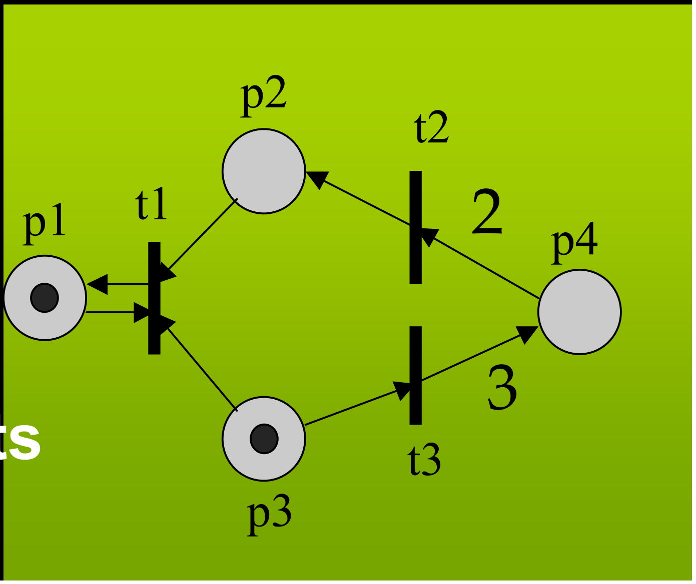
# Petri Net Graph

- **Bipartite weighted directed graph:**
  - **Places: circles**
  - **Transitions: bars or boxes**
  - **Arcs: arrows labeled with weights**
- **Tokens: black dots**



# Petri Net

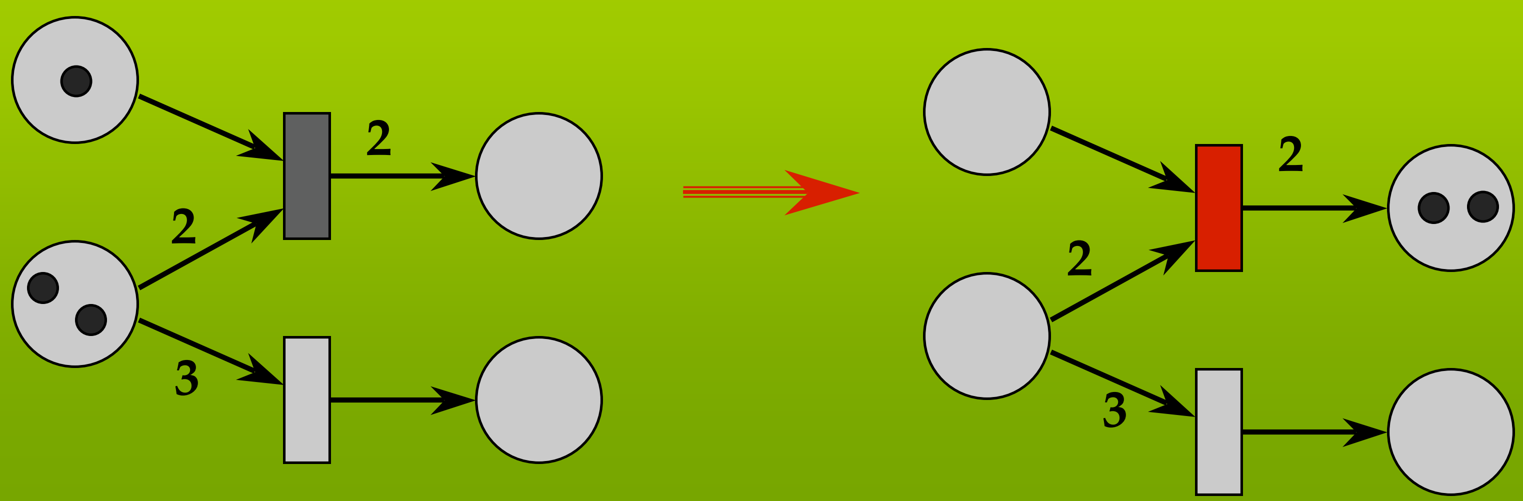
- **A PN (N,M0) is a Petri Net Graph N**
  - **places: represent distributed state by holding tokens**
    - **marking (state) M is an n-vector ($m_1$,$m_2$,$m_3$…), where $m_i$ is the non-negative number of tokens in place $p_i$.**
    - **initial marking ($M_0$) is initial state**
  - **transitions: represent actions/events**
    - **enabled transition: enough tokens in predecessors**
    - **firing transition: modifies marking**
- **…and an initial marking M0.**



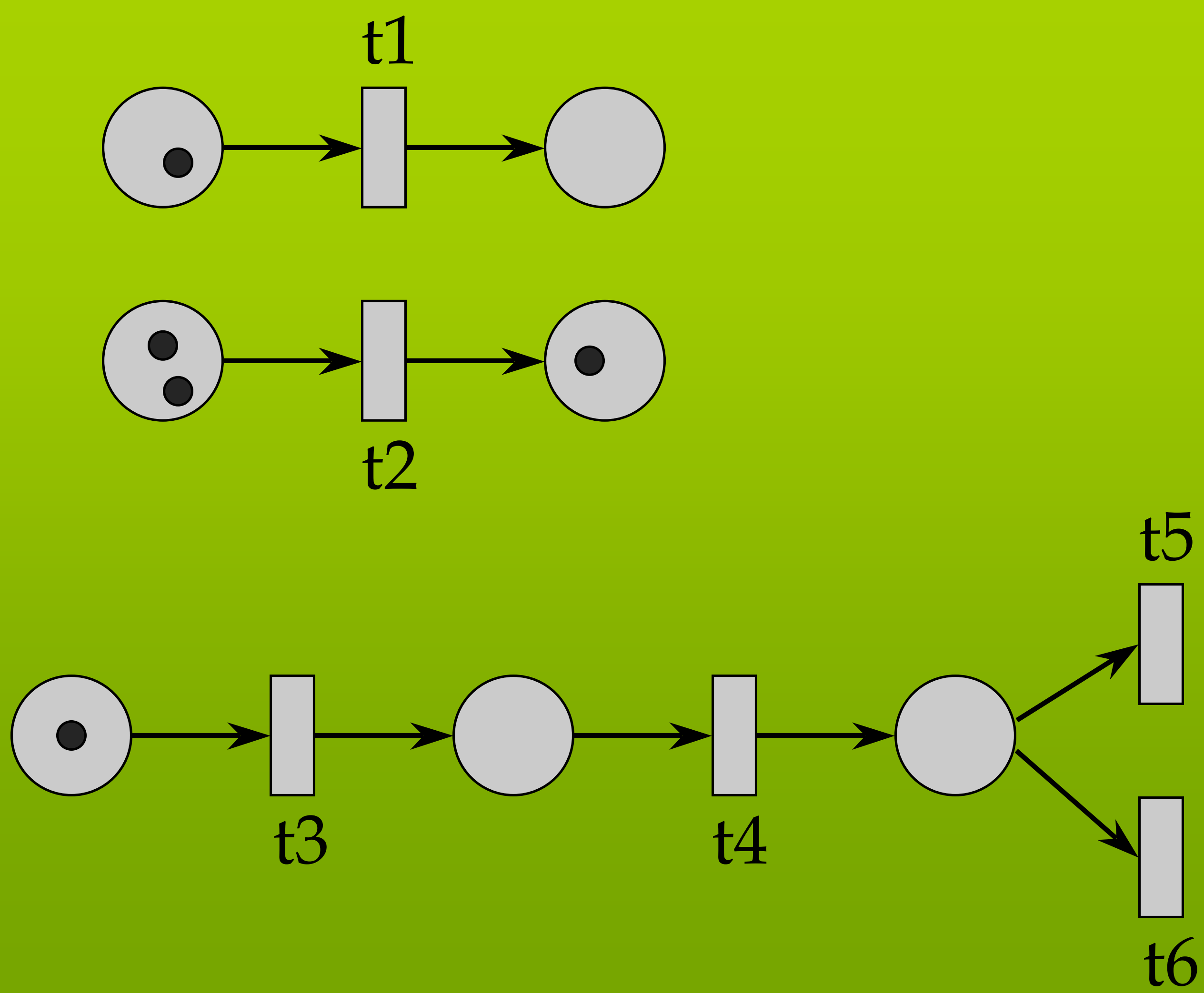

**Places/Transitions: conditions/events**

# Transition firing rule

- **A marking is changed according to the following rules:**
  - **A transition is enabled if there are enough tokens in each input place**
  - **An enabled transition may or may not fire**
  - **The firing of a transition modifies marking by consuming tokens from the input places and producing tokens in the output places**



# Concurrency, causality, choice



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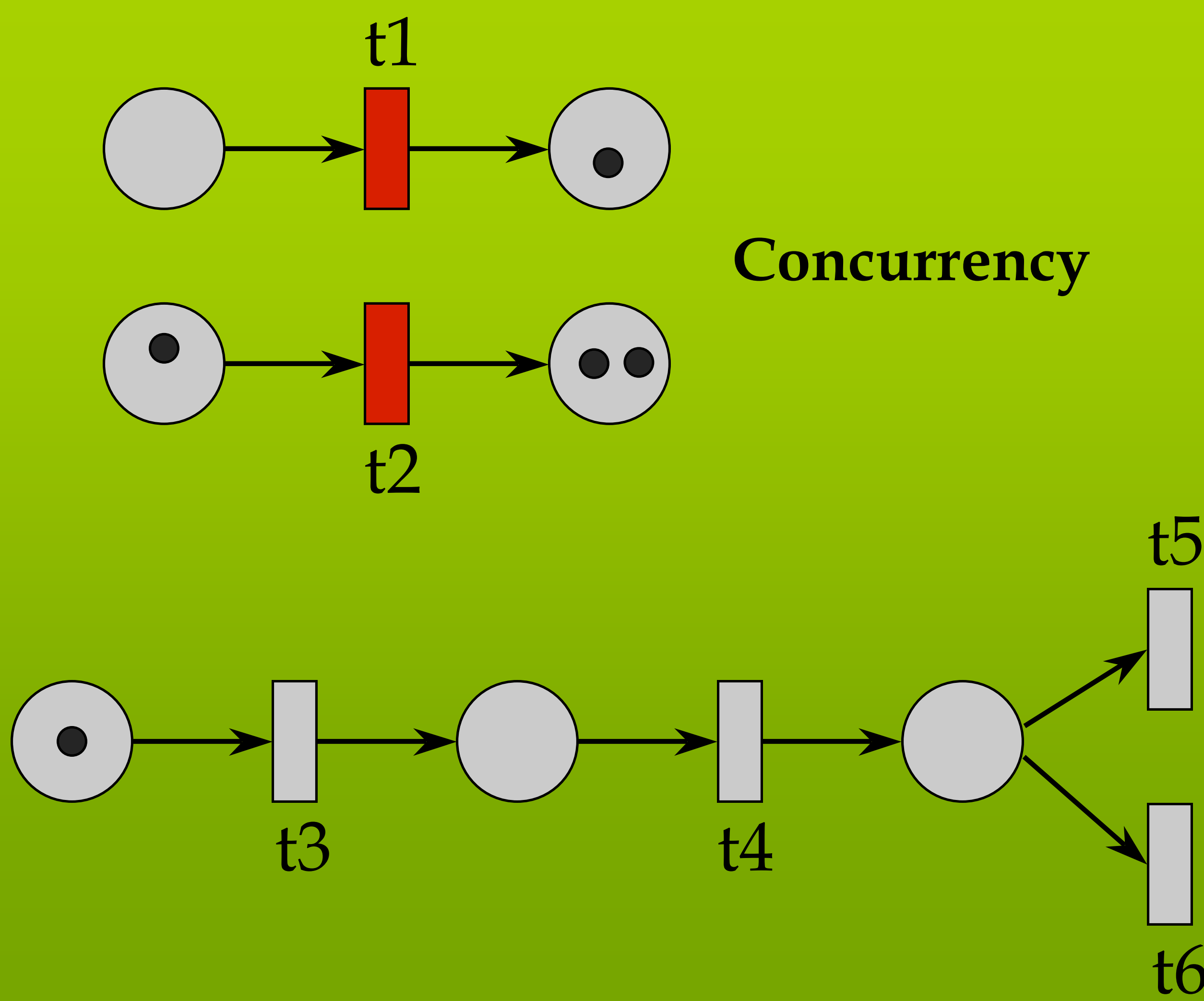

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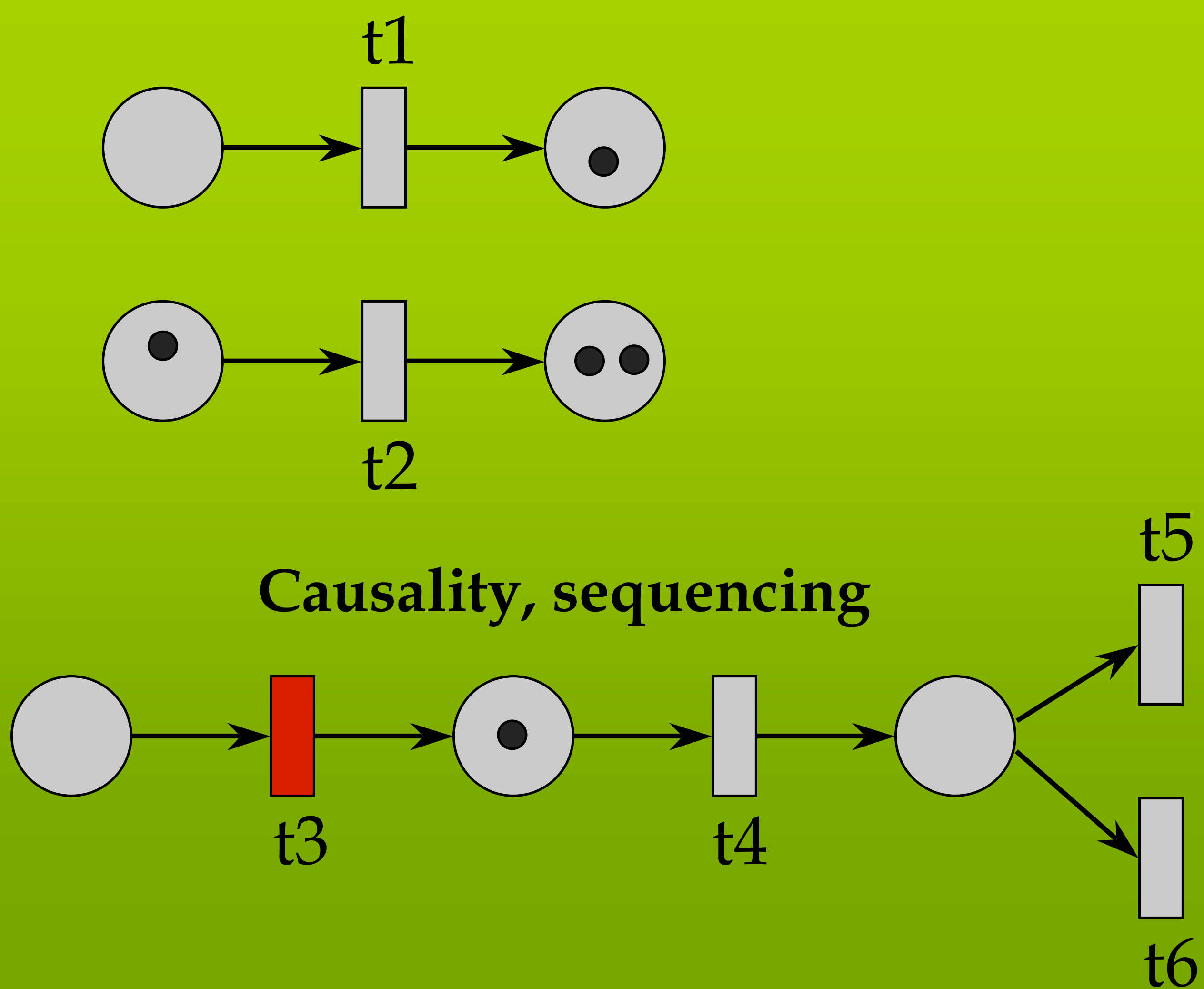

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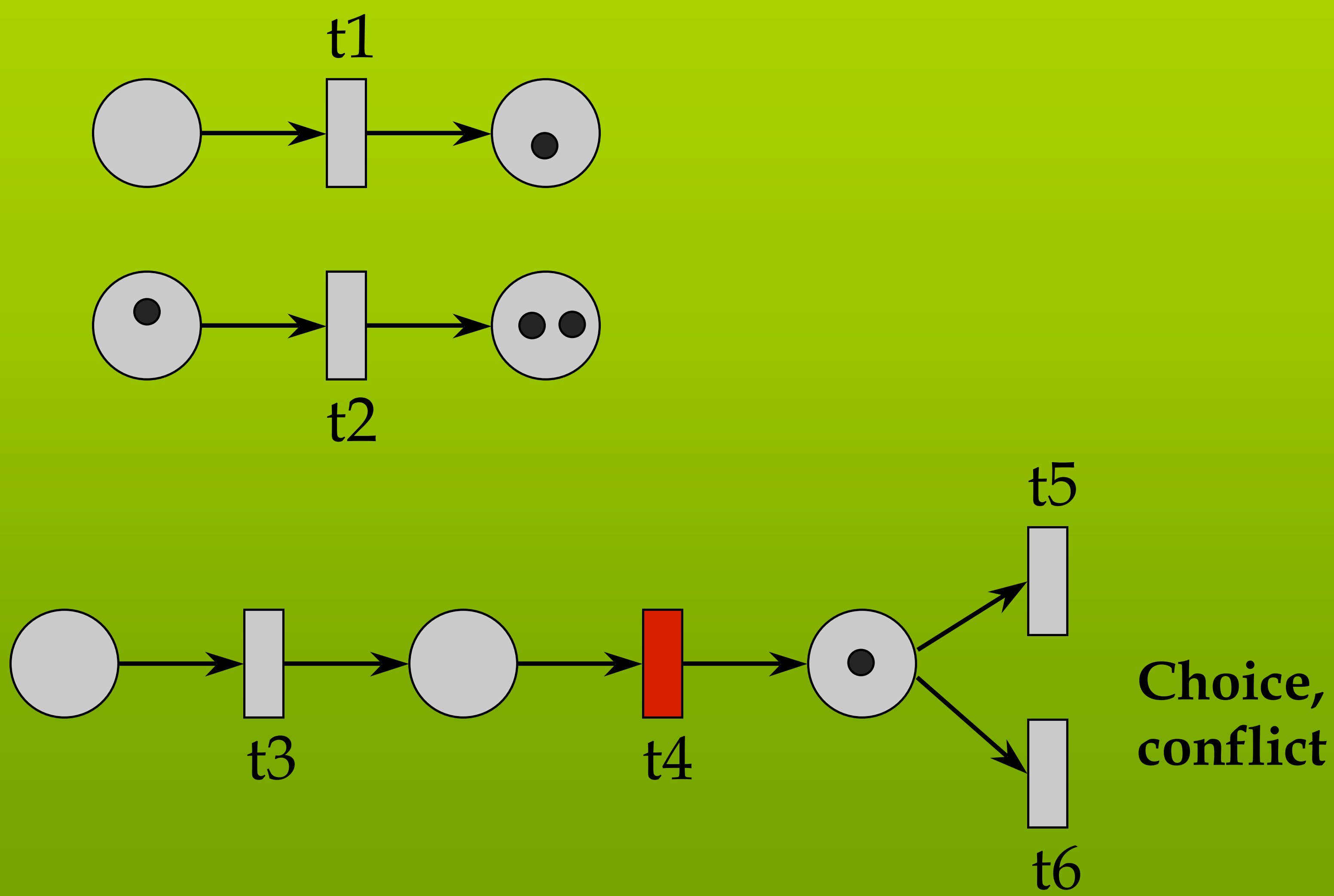

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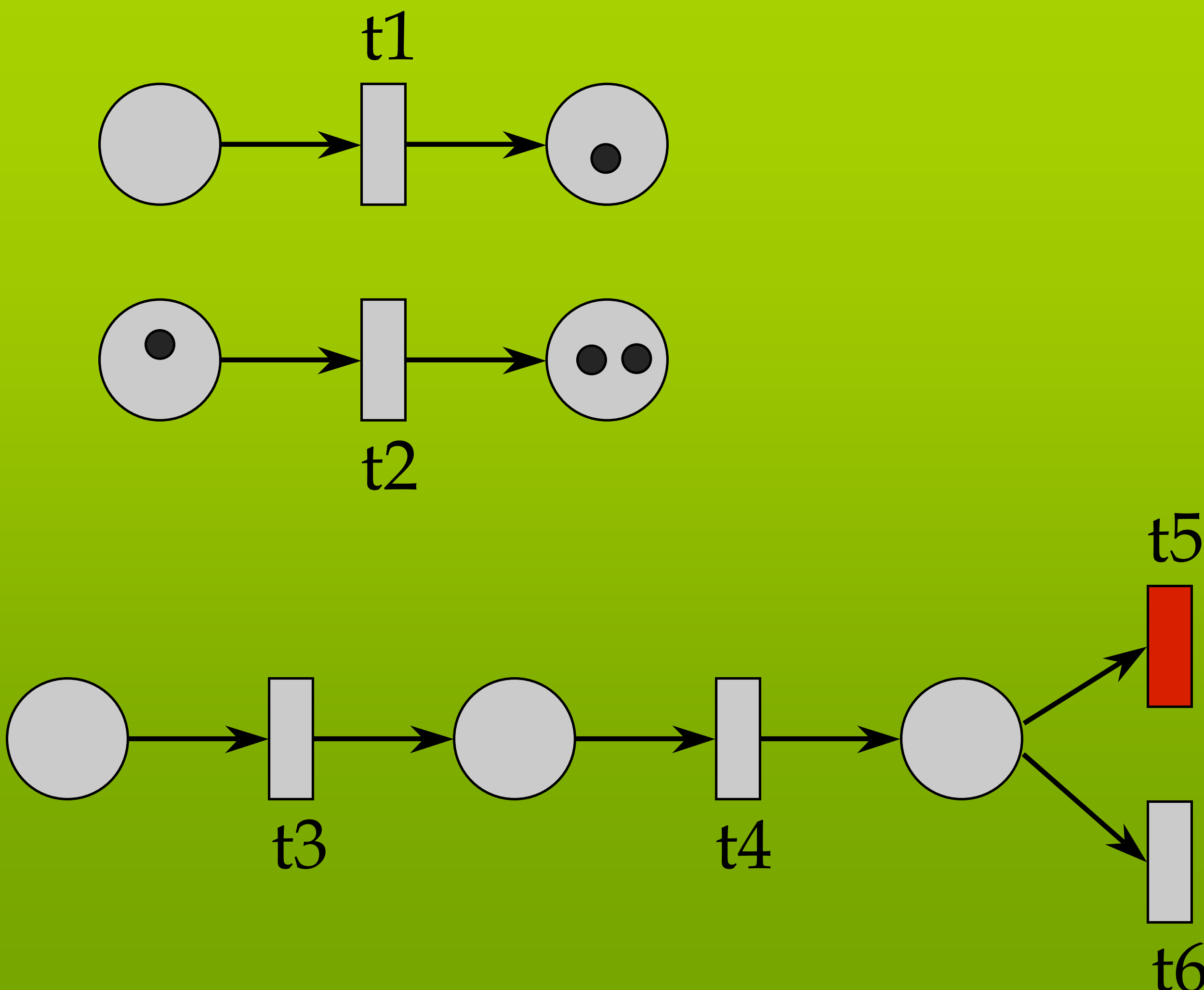
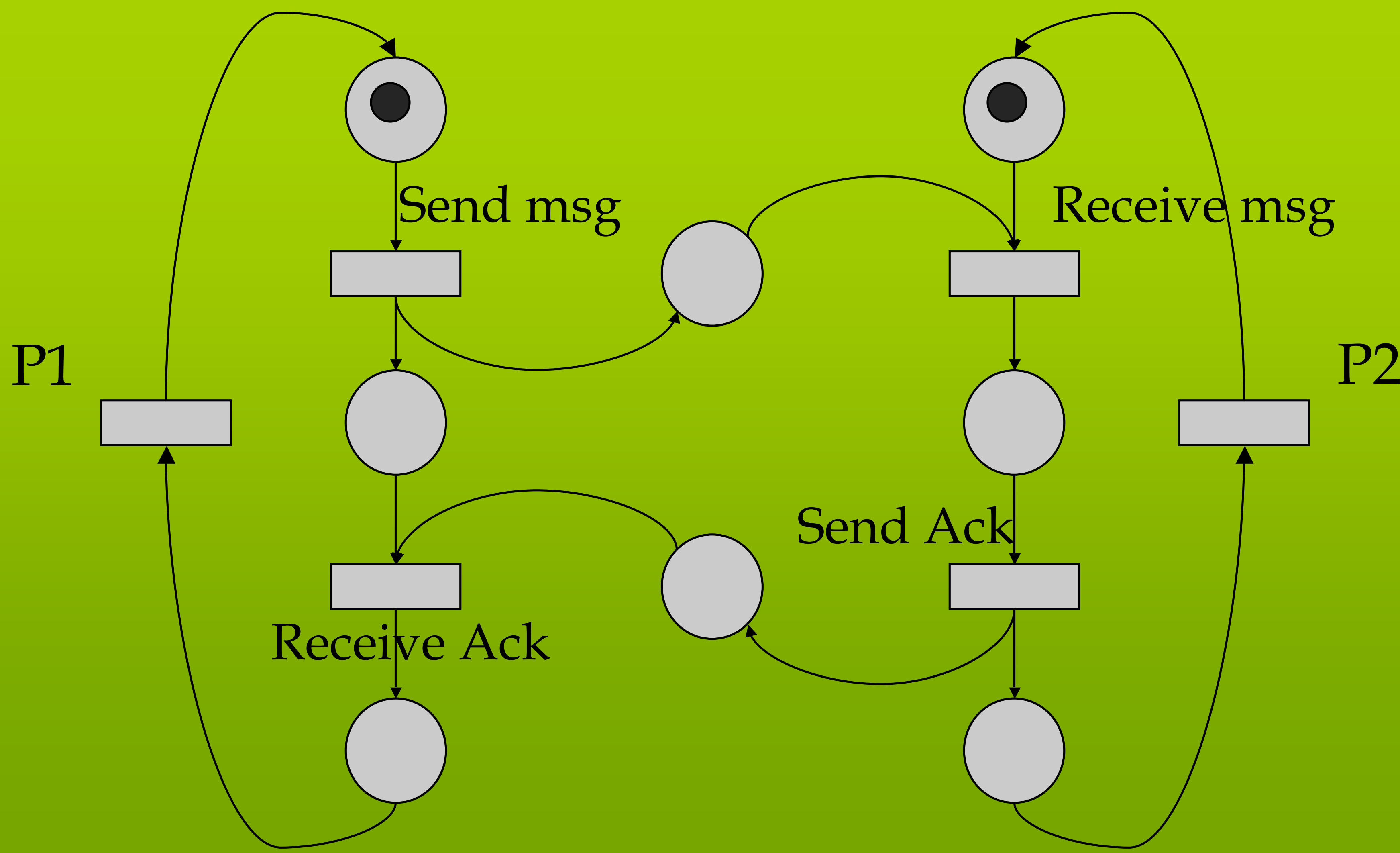



**Choice, conflict**

# Communication Protocol



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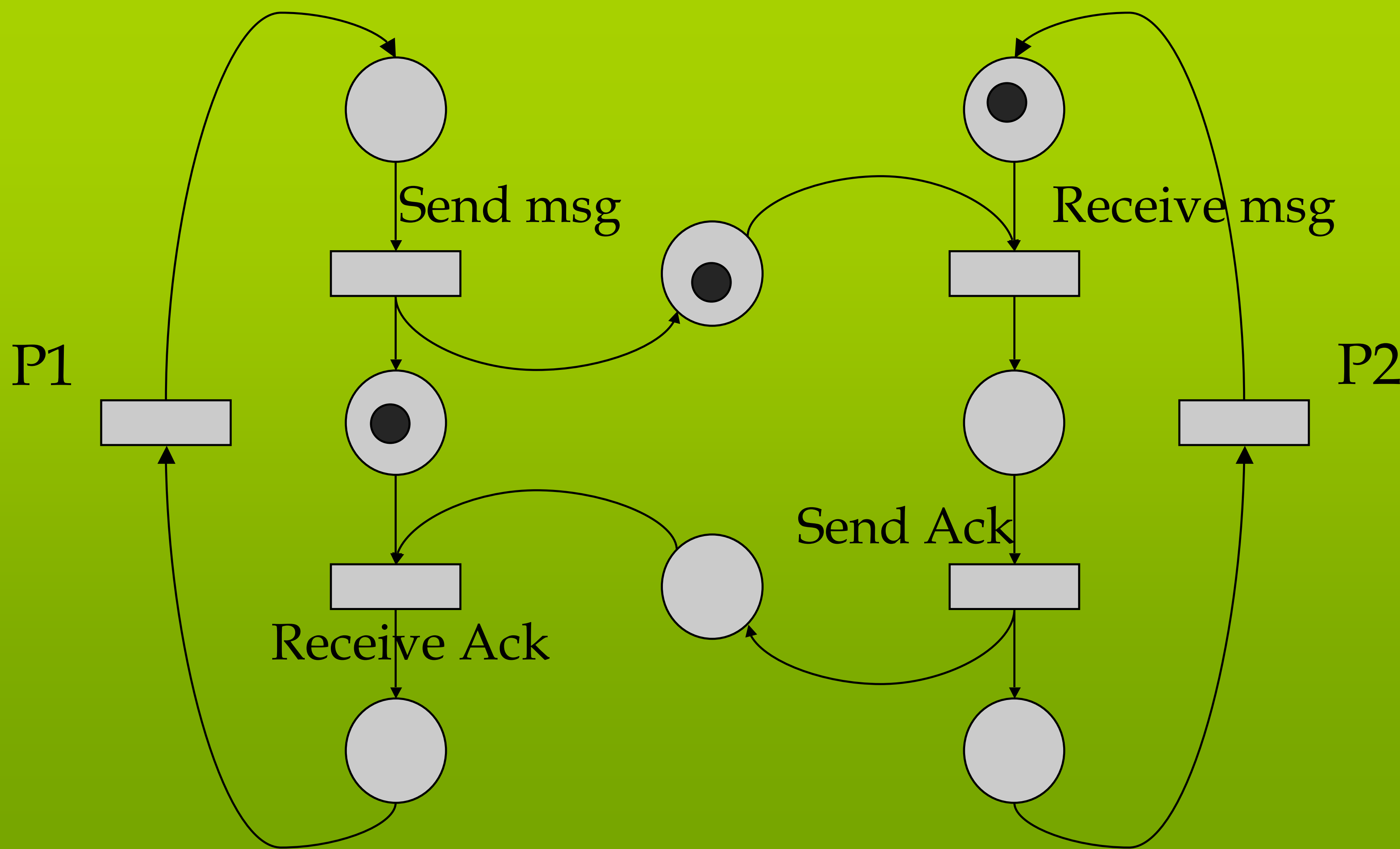

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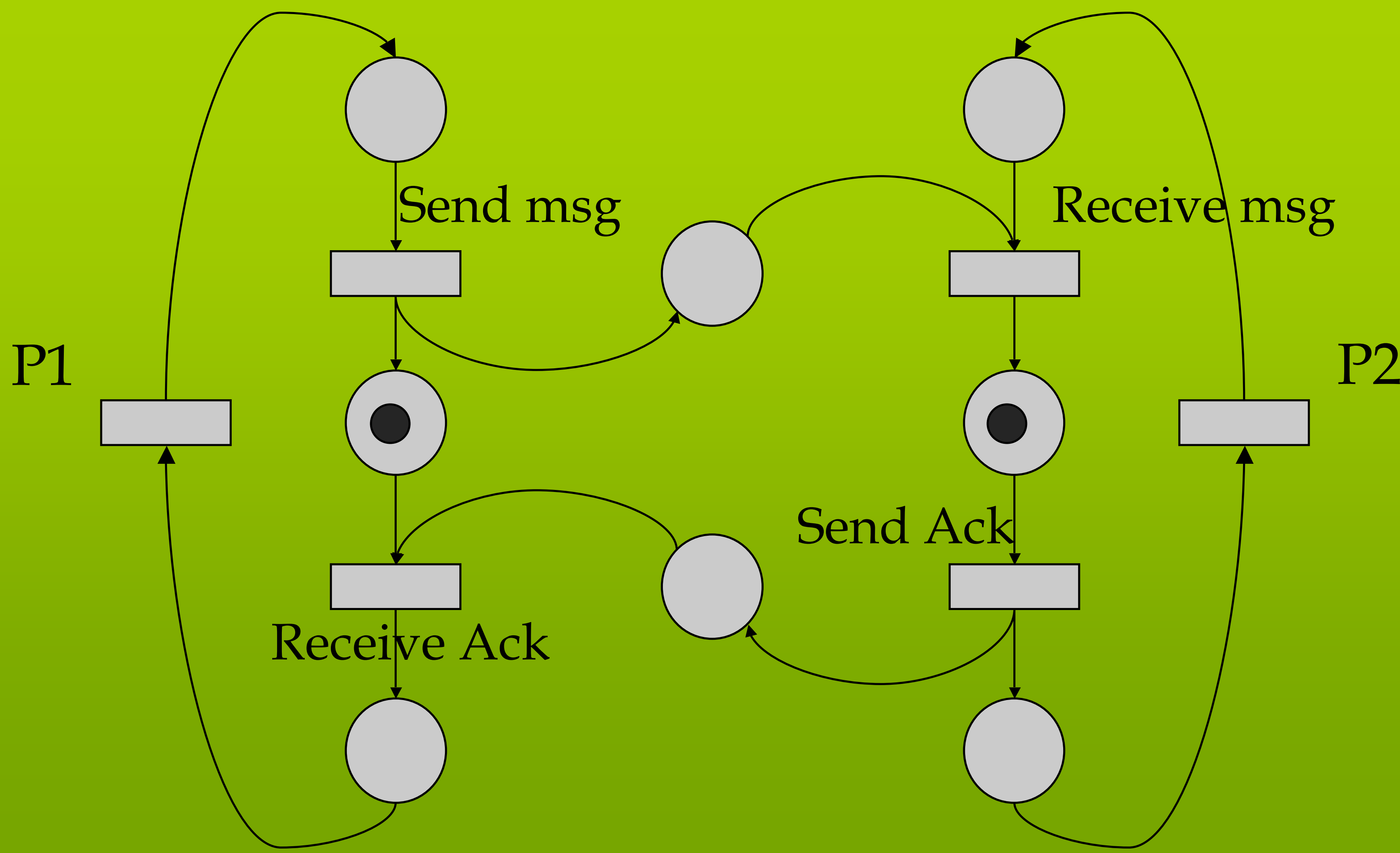

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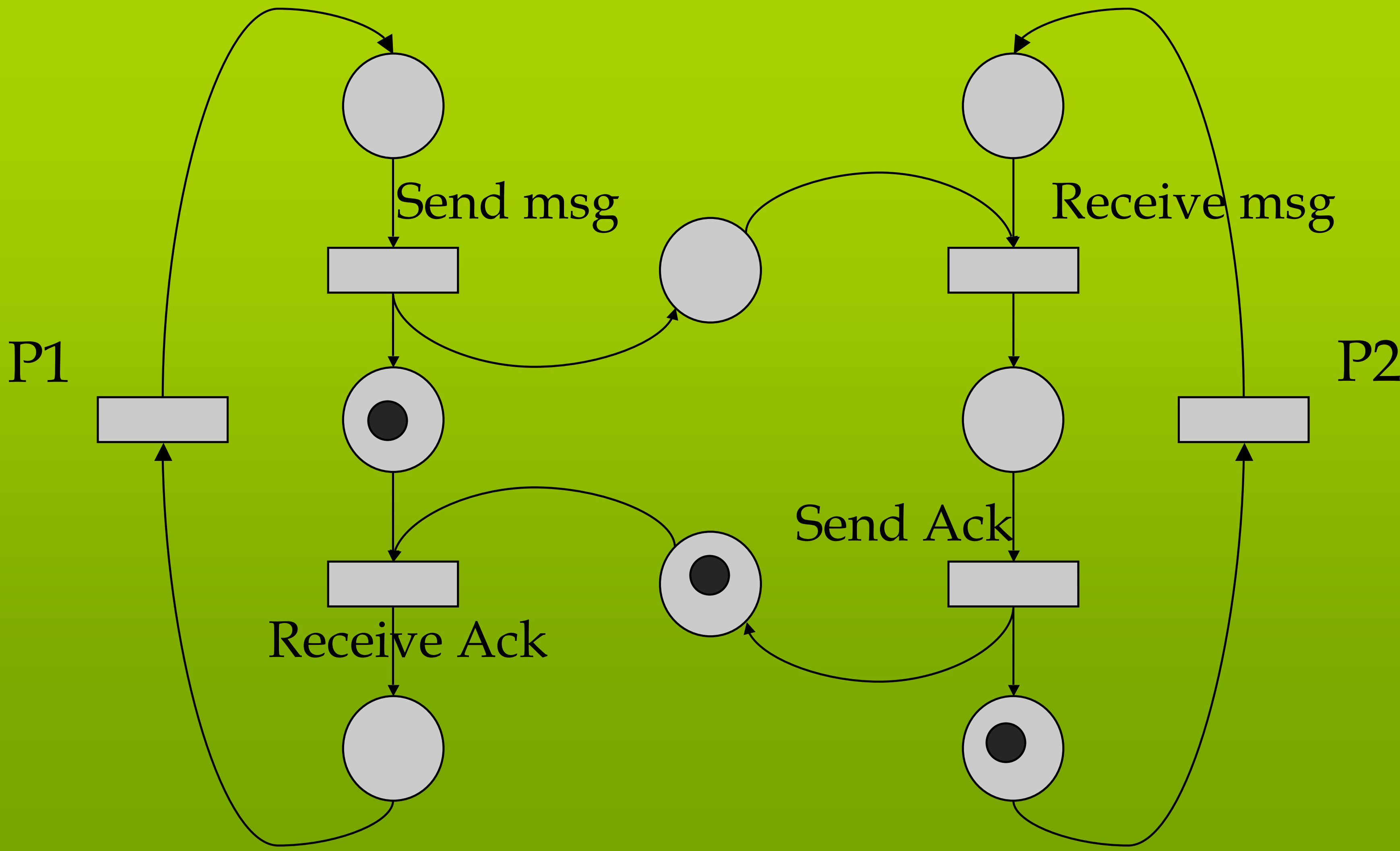

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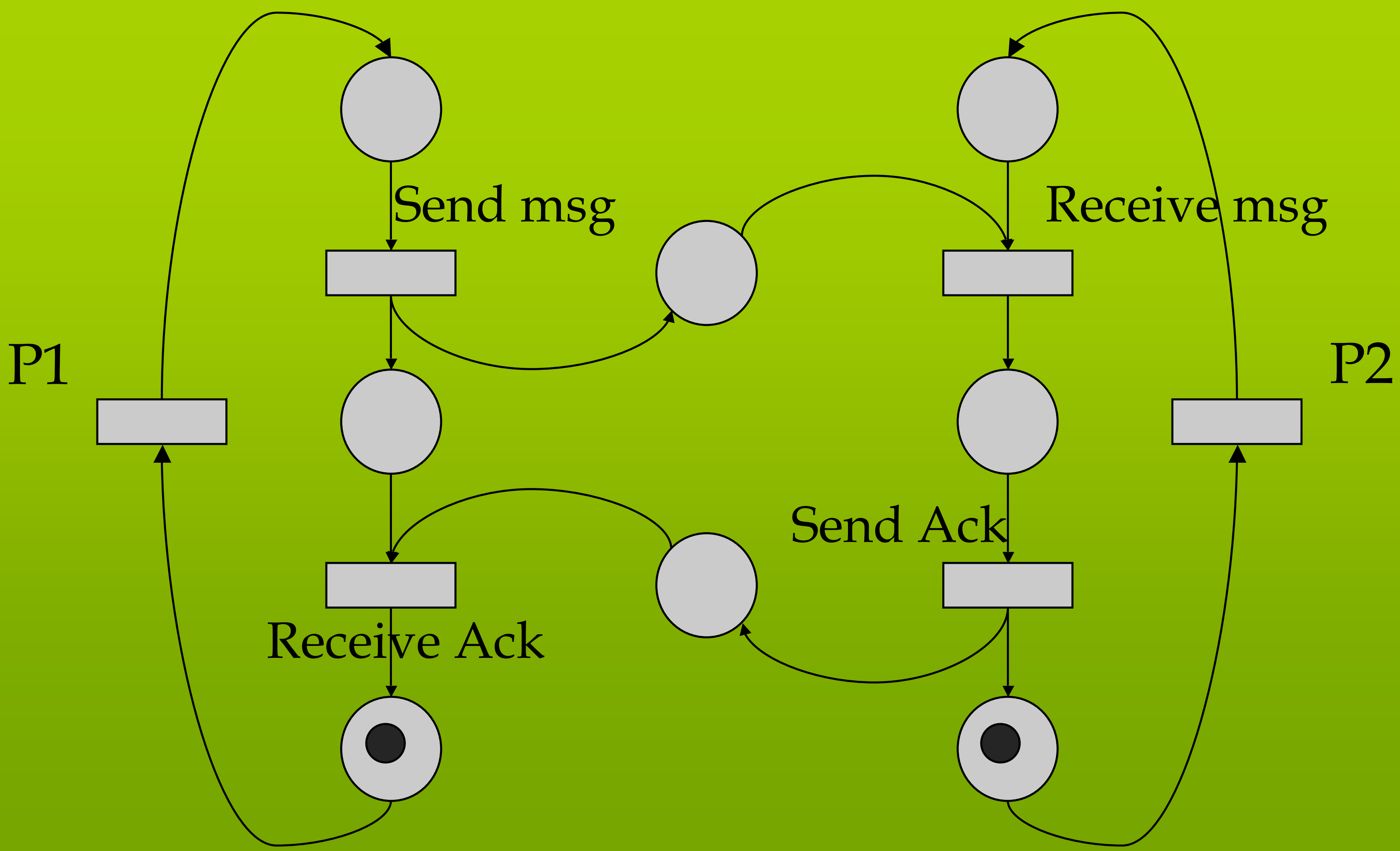

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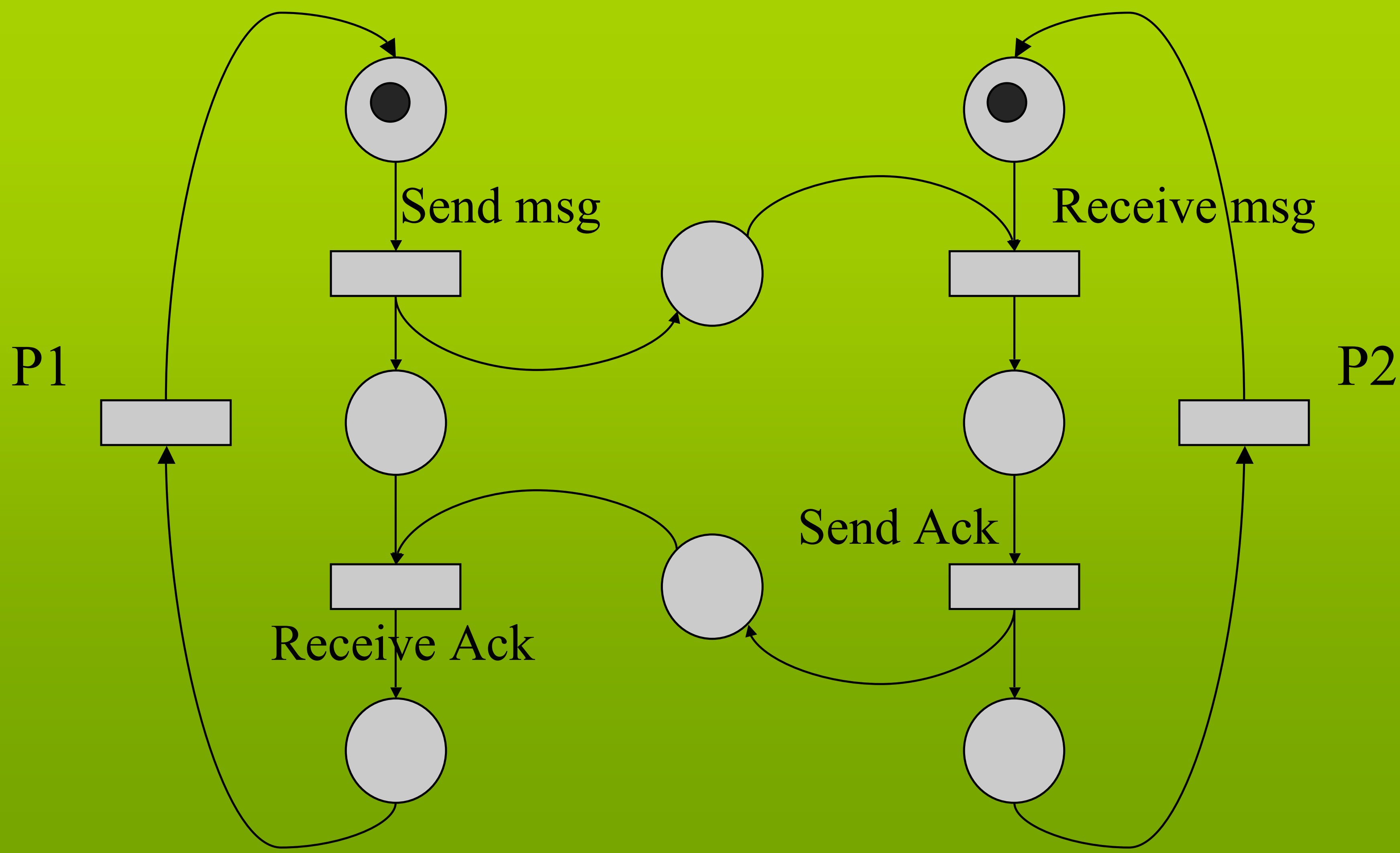
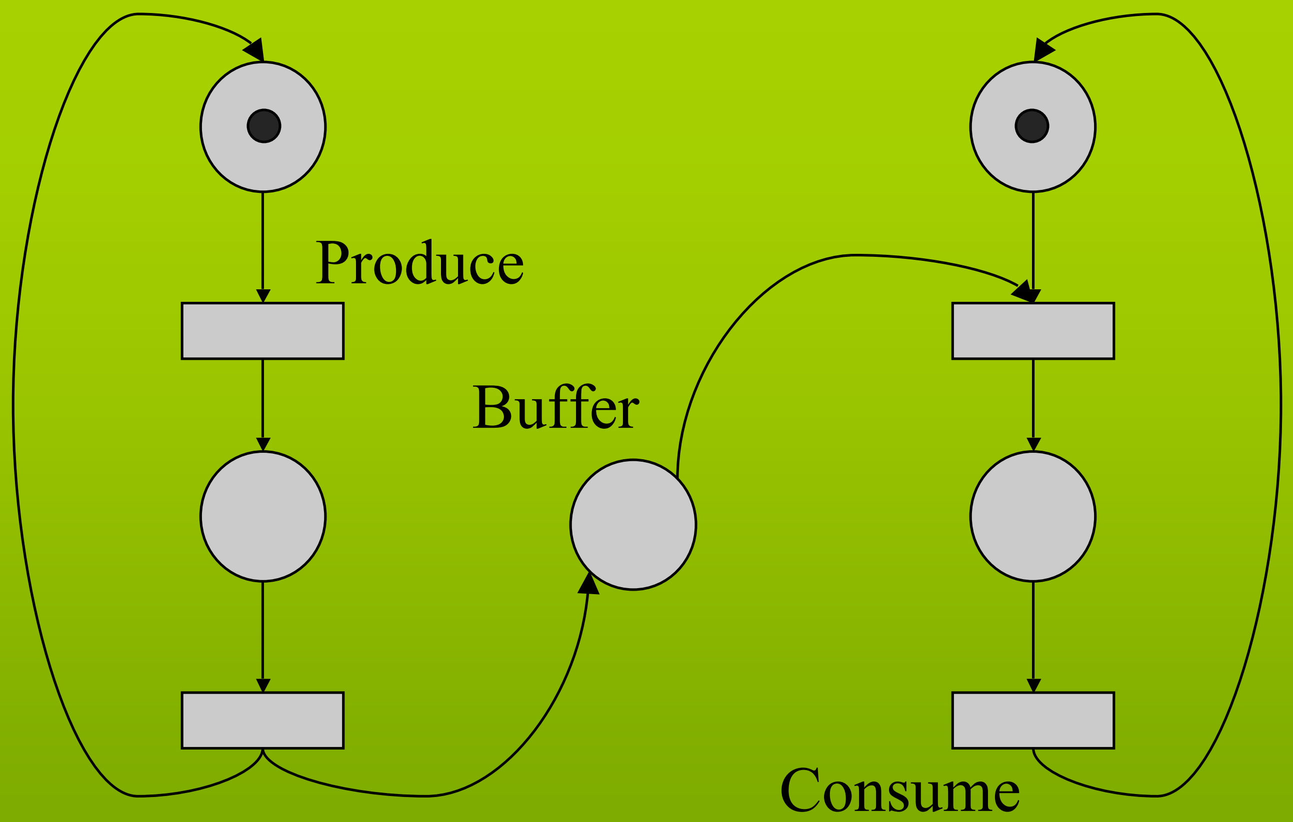


# Producer-Consumer Problem



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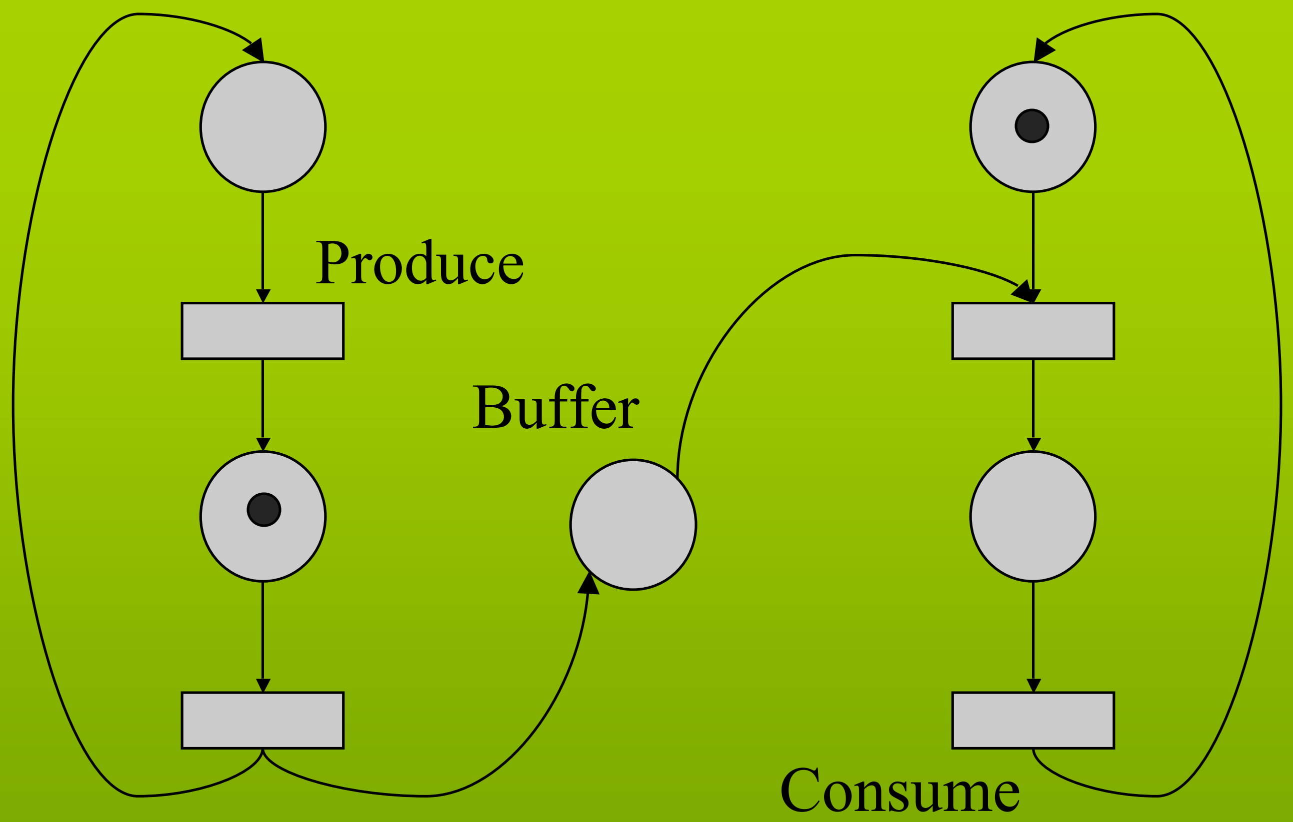

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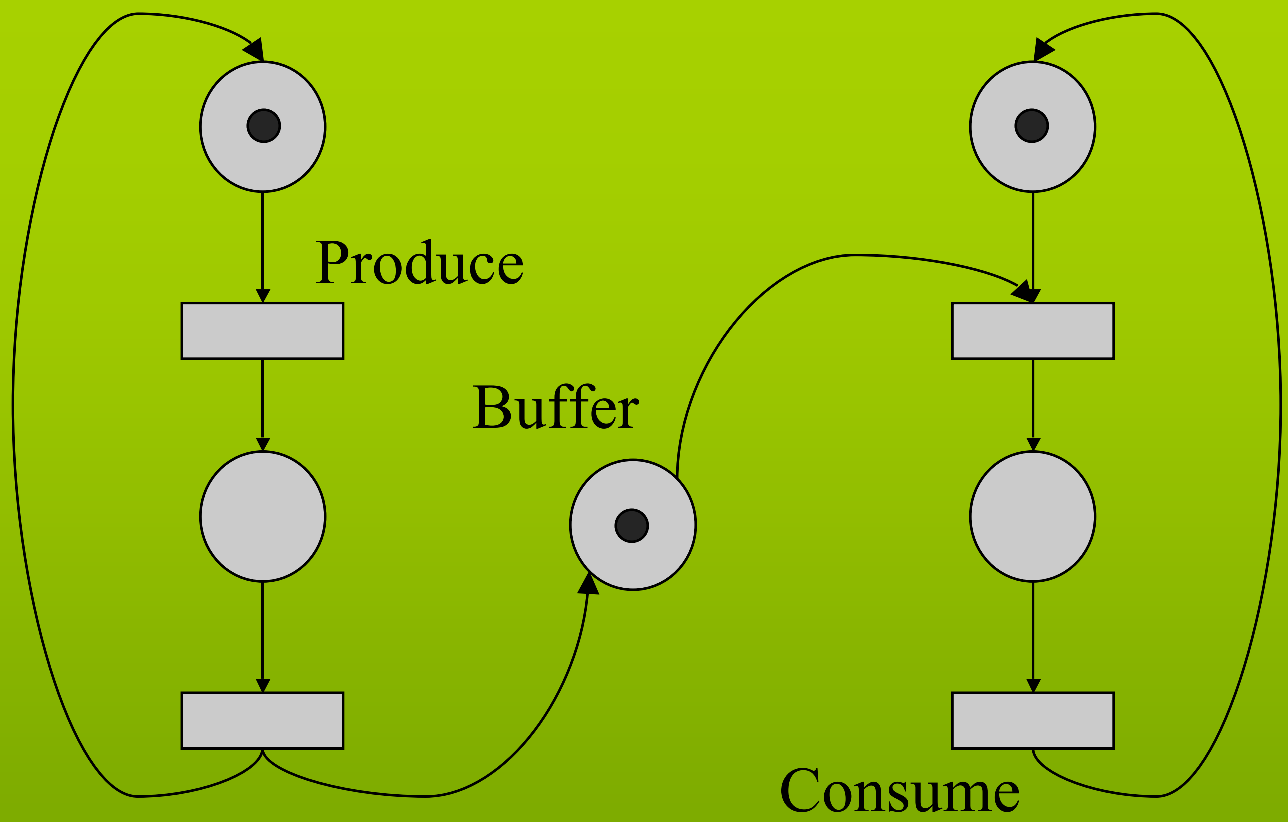

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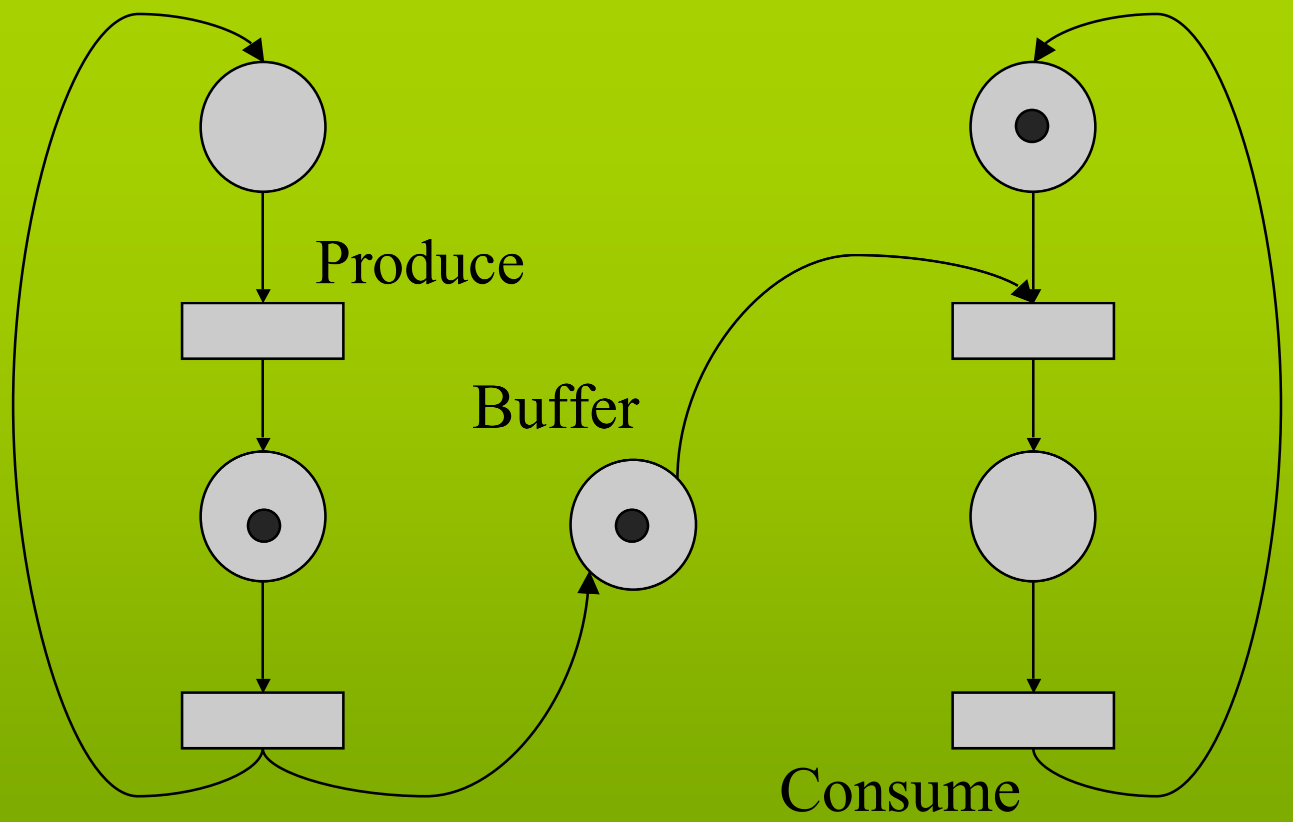

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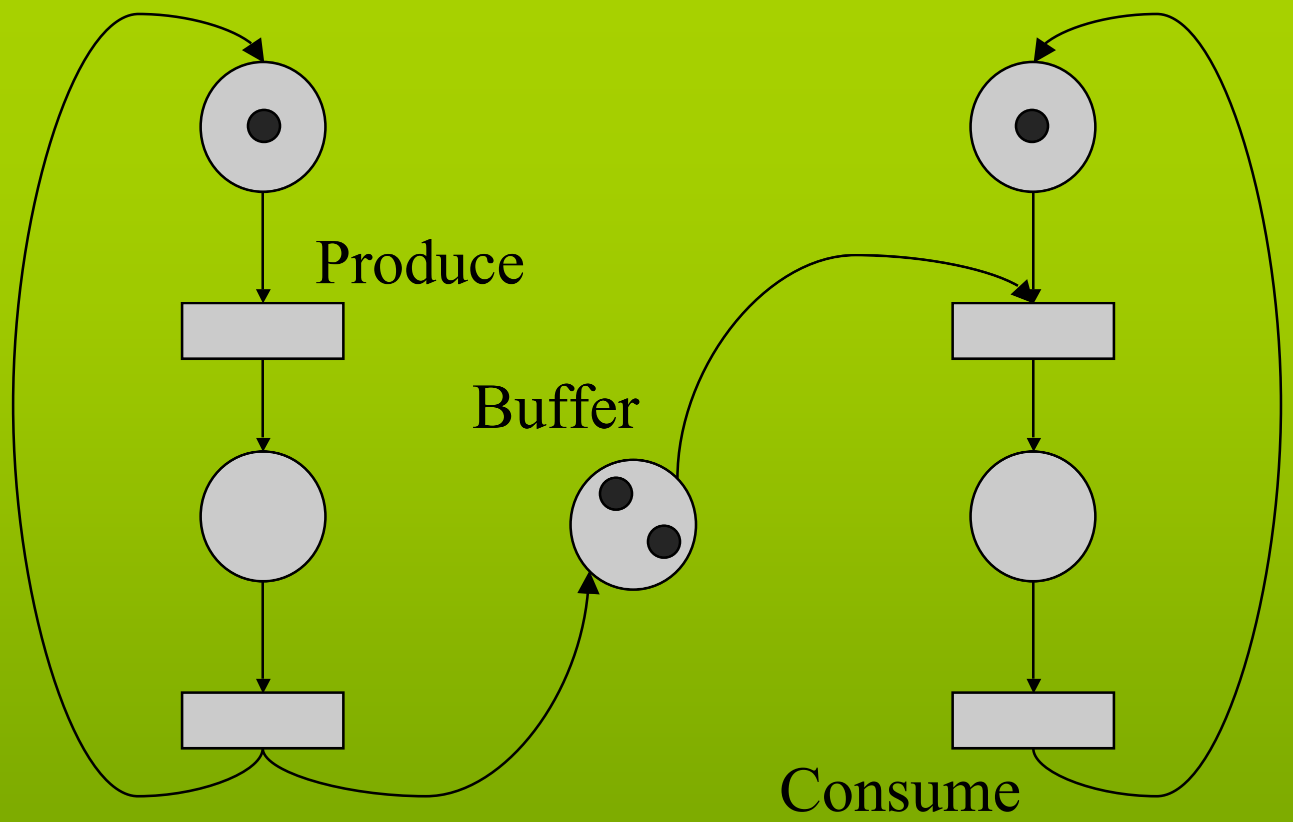

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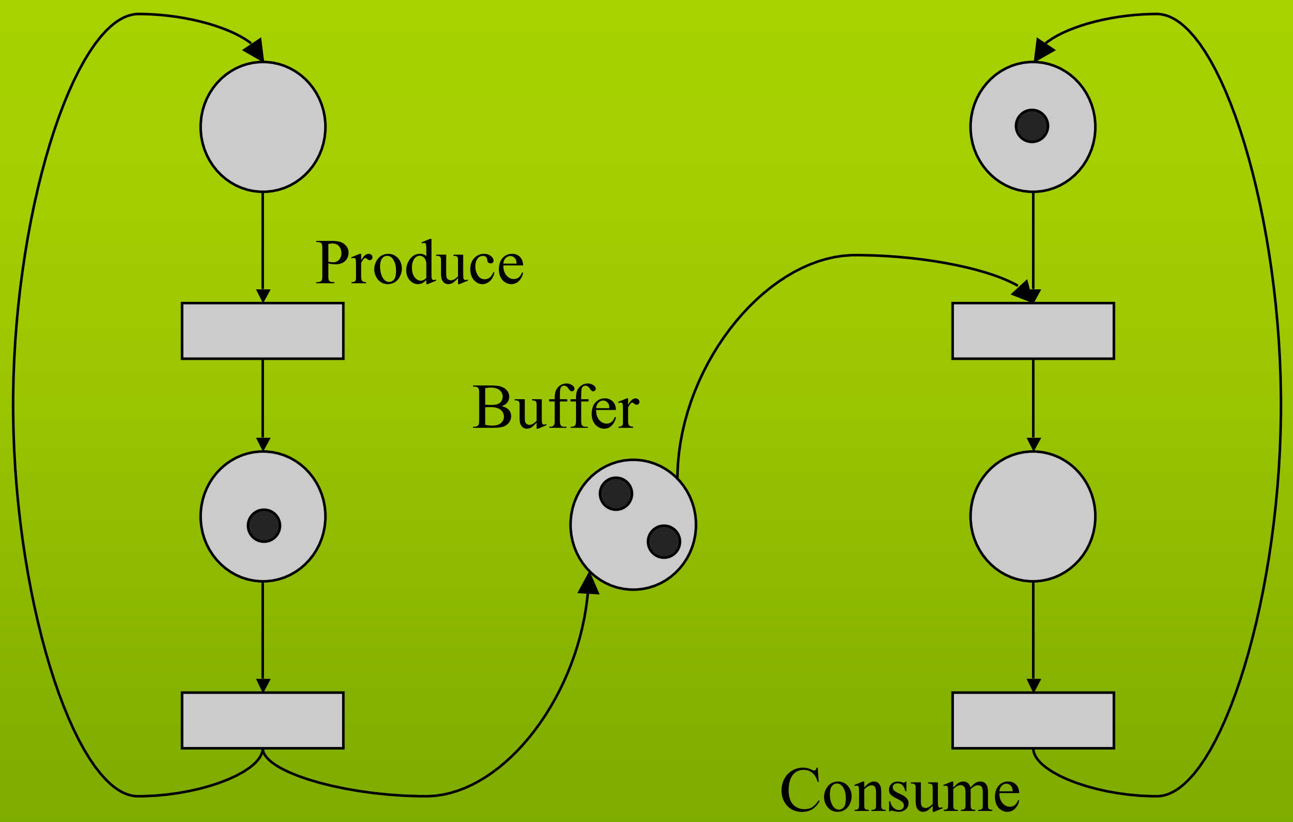

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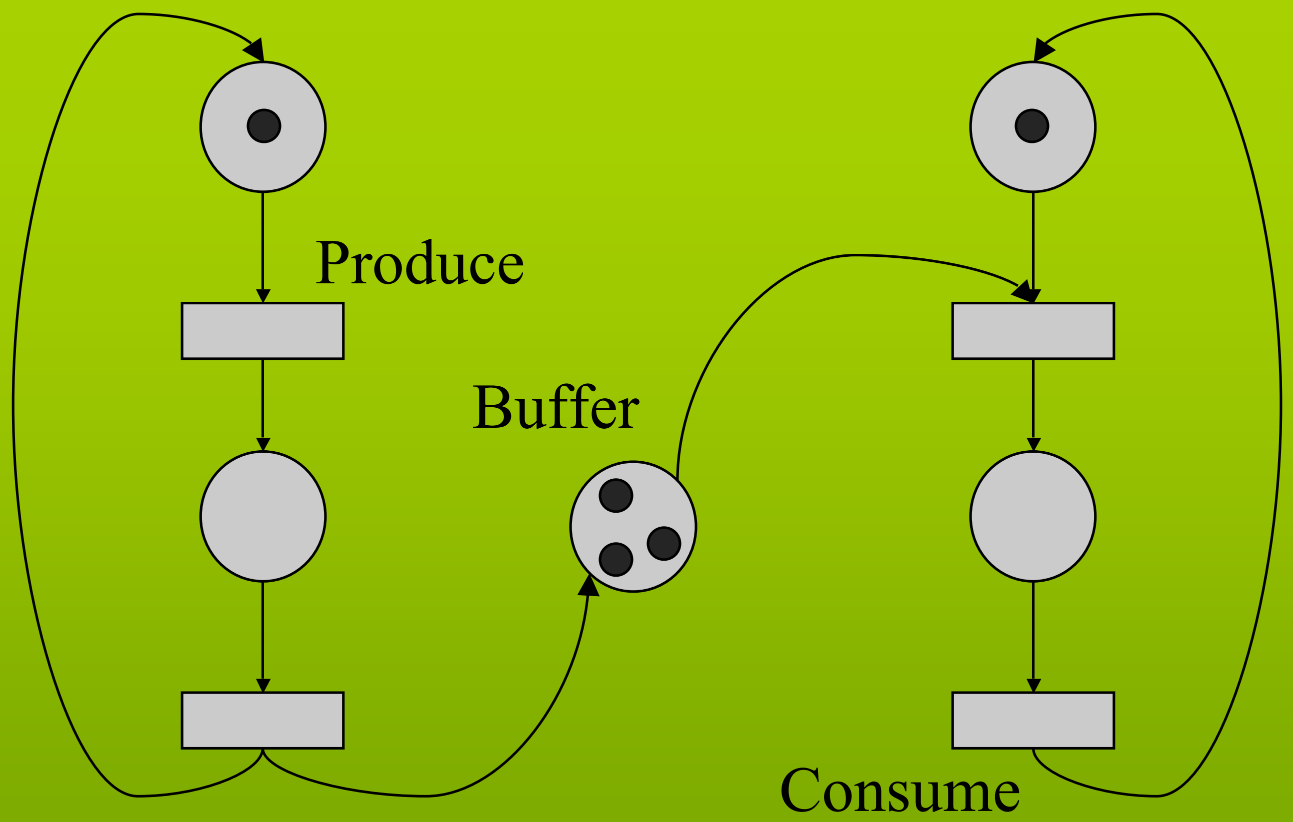

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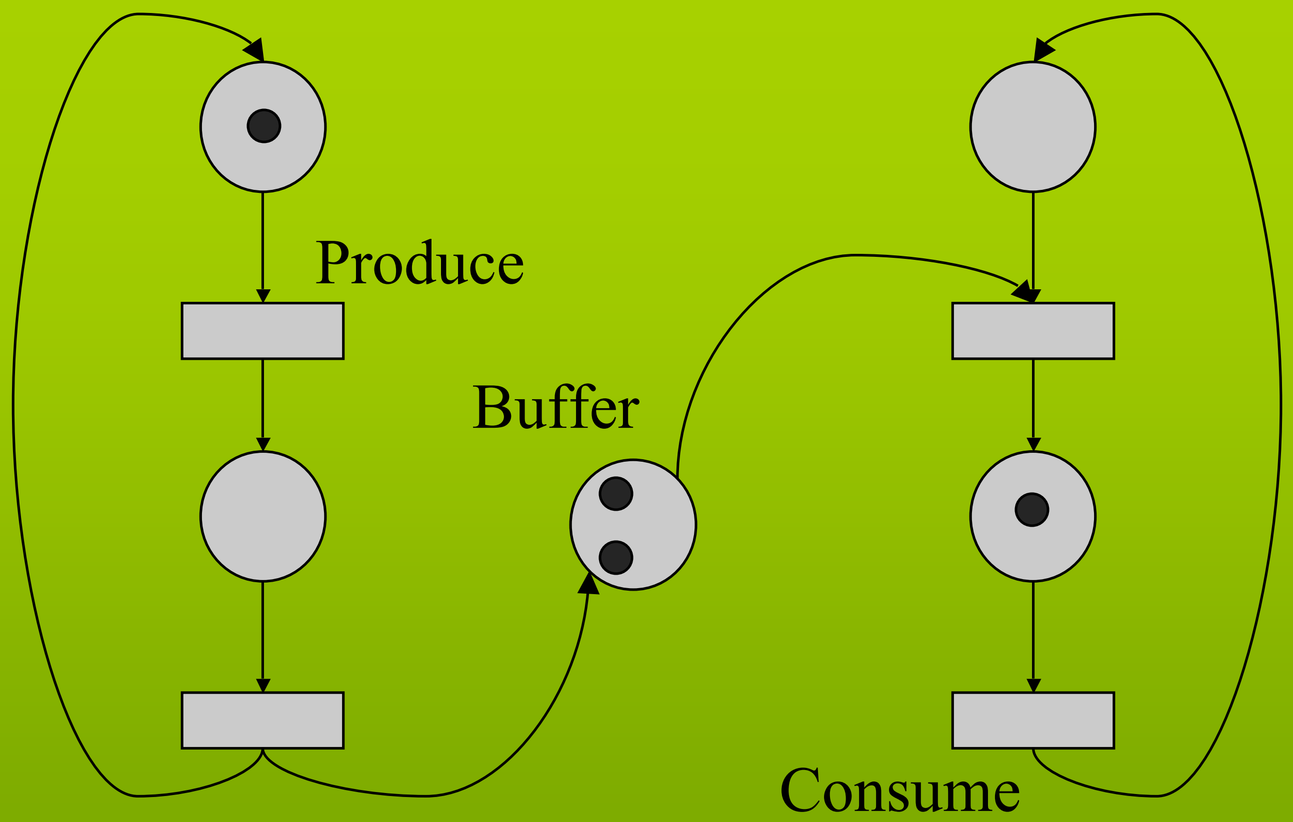

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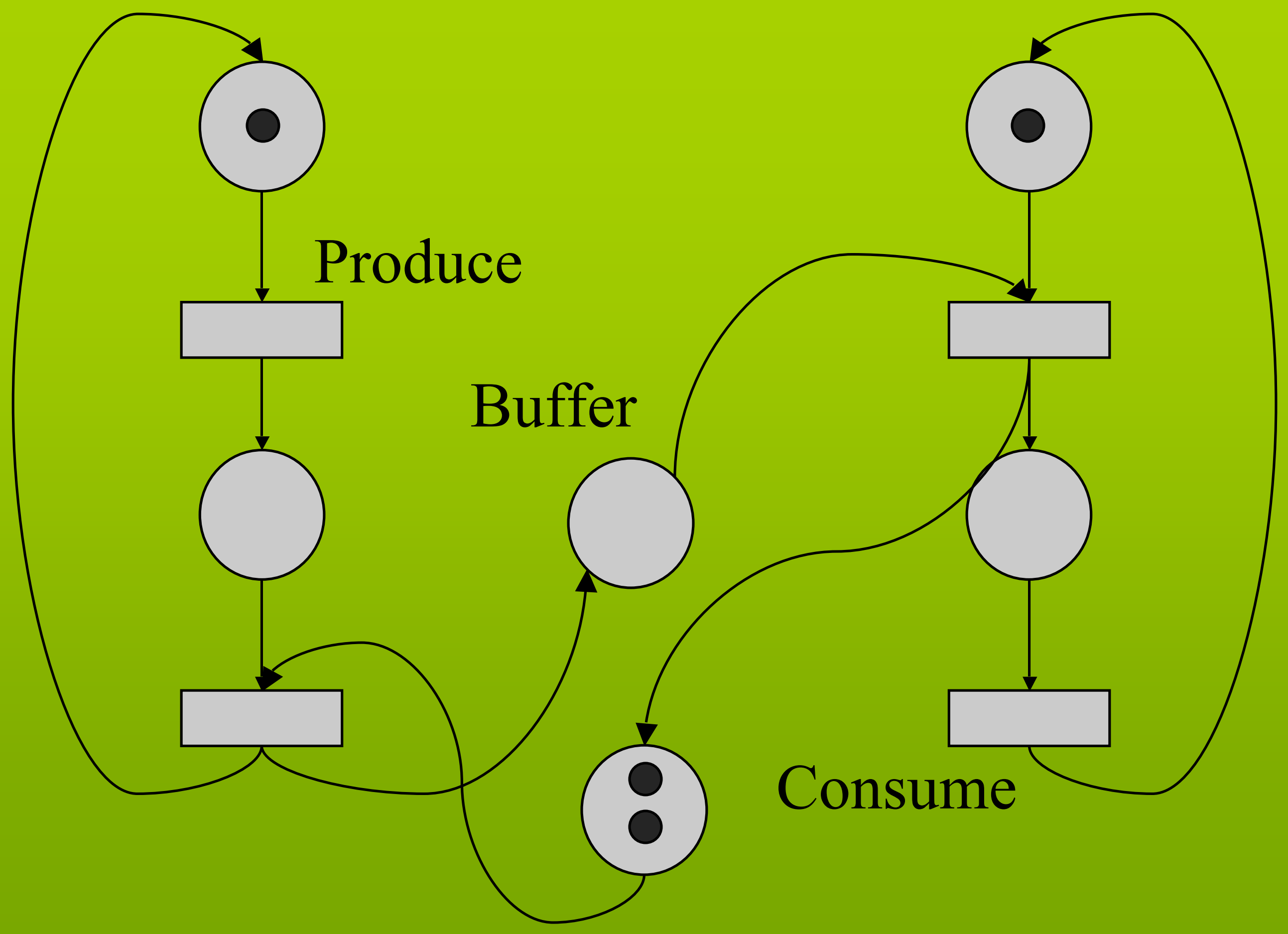

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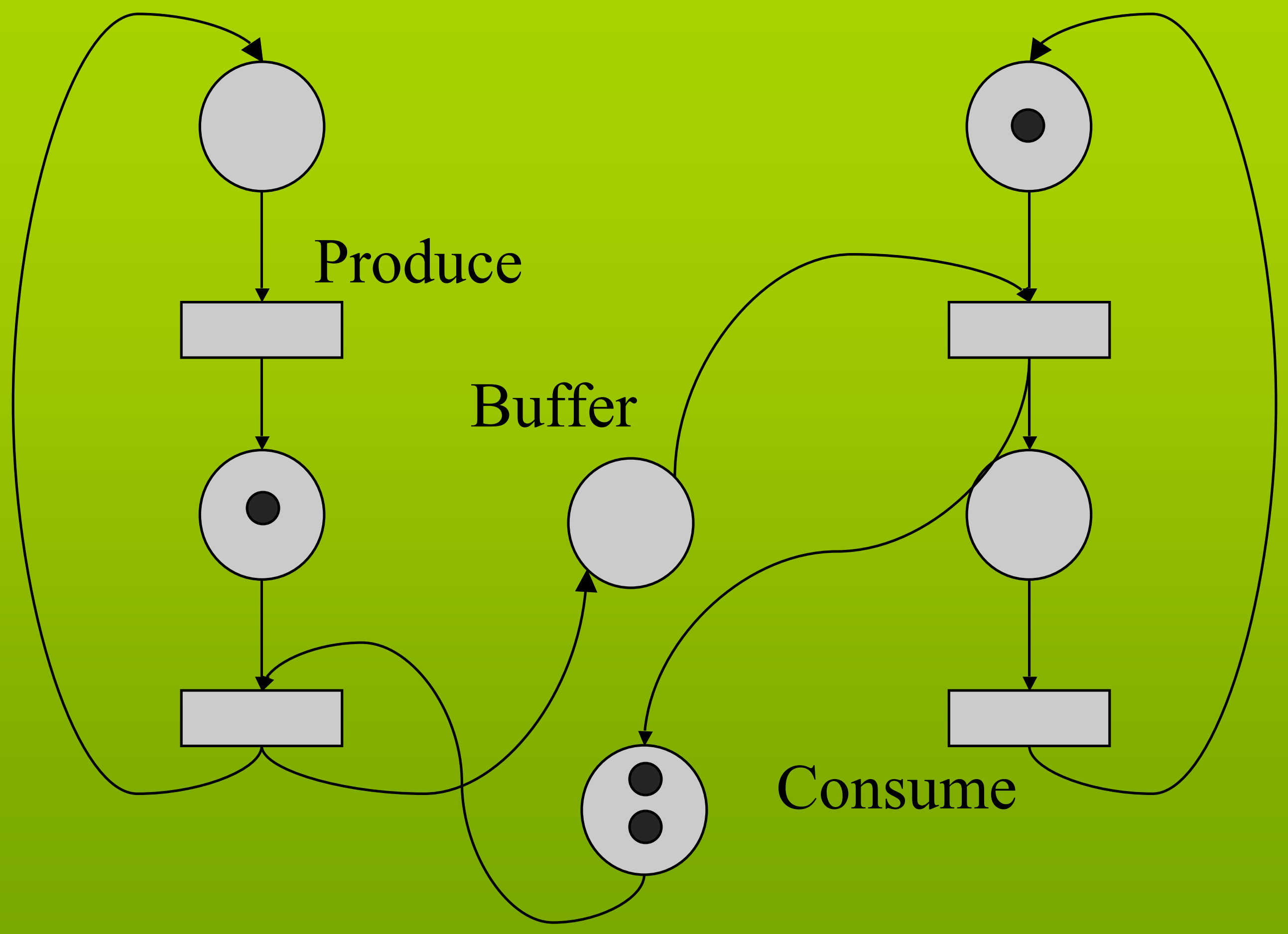

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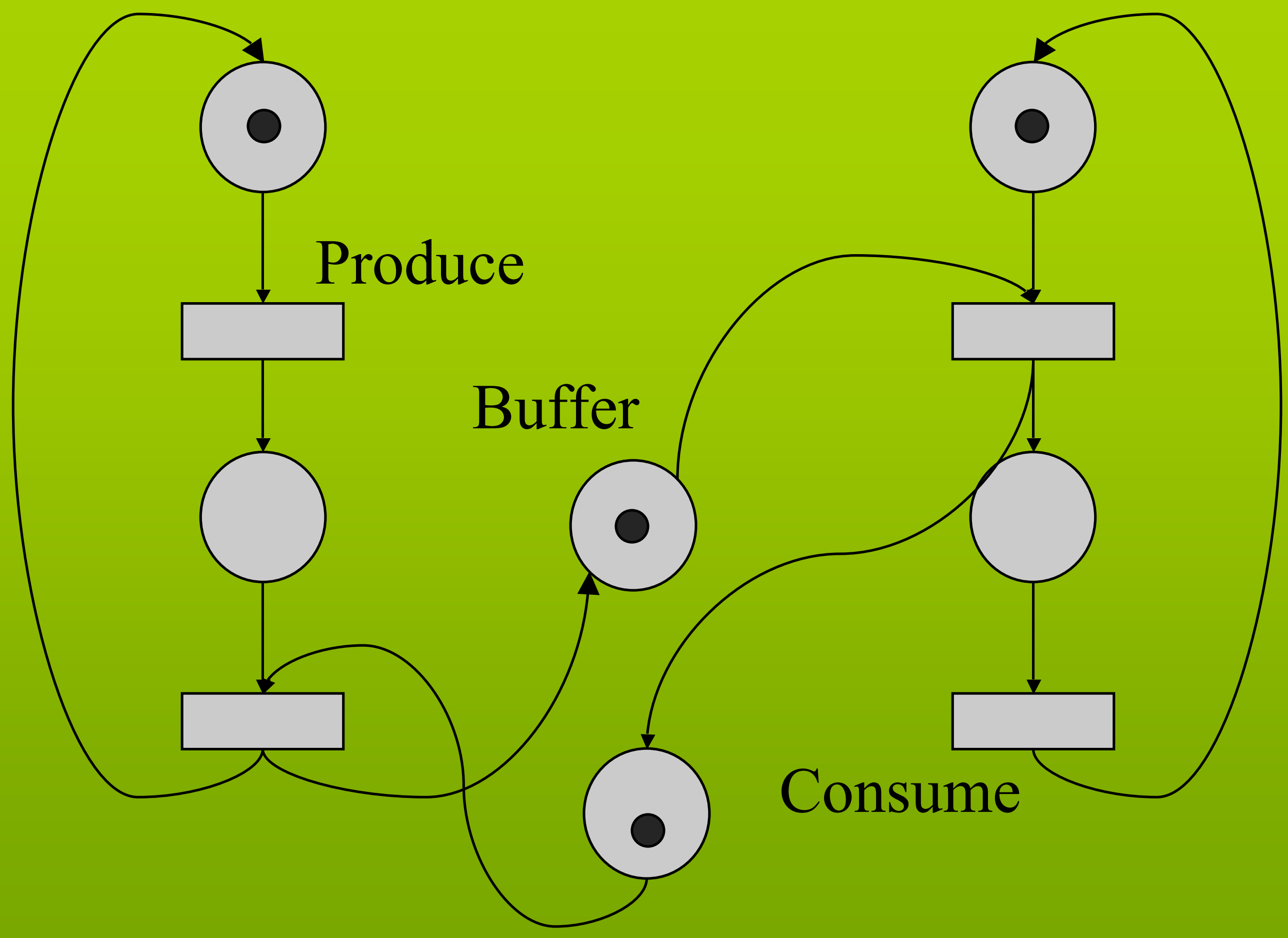

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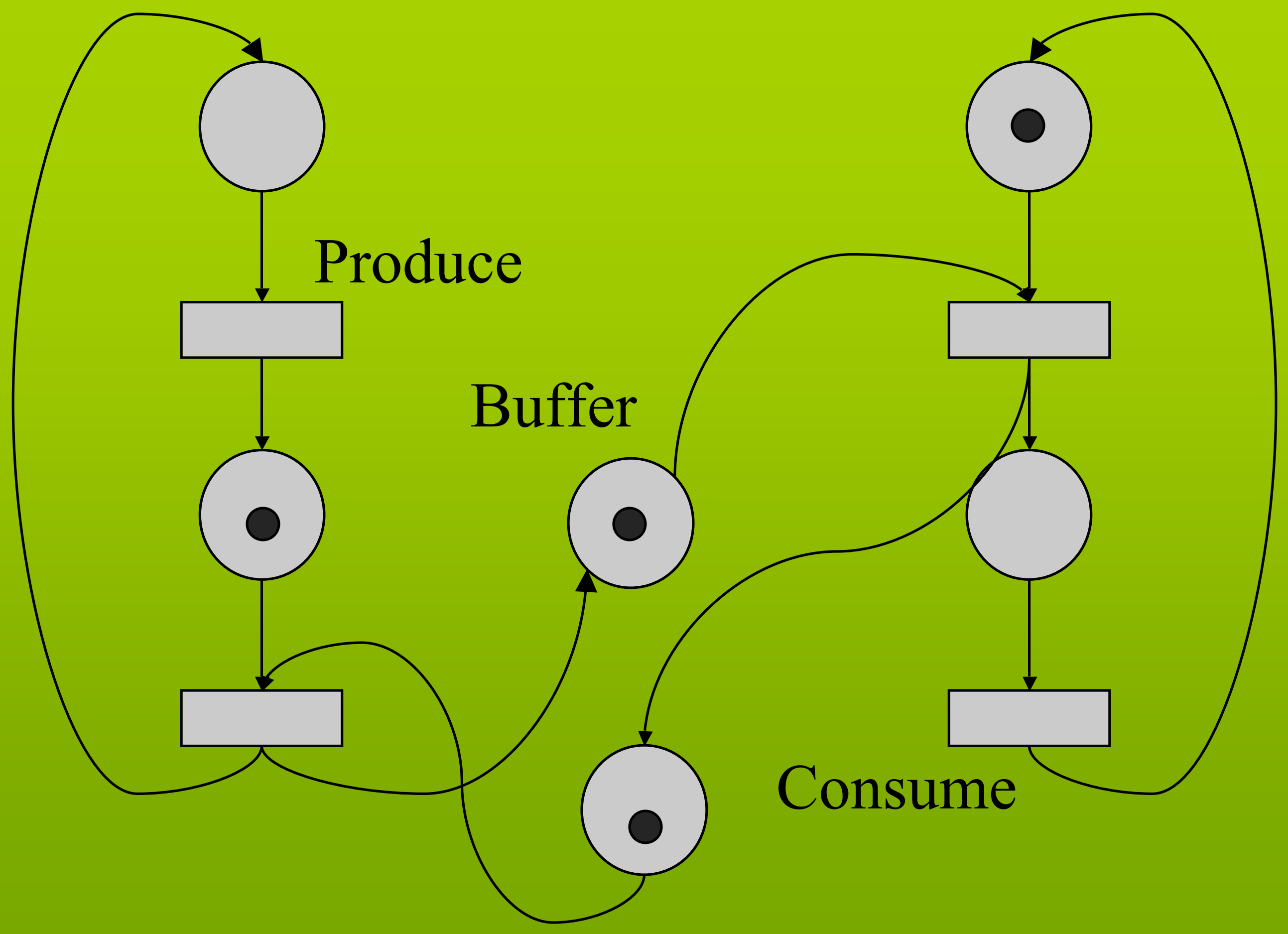

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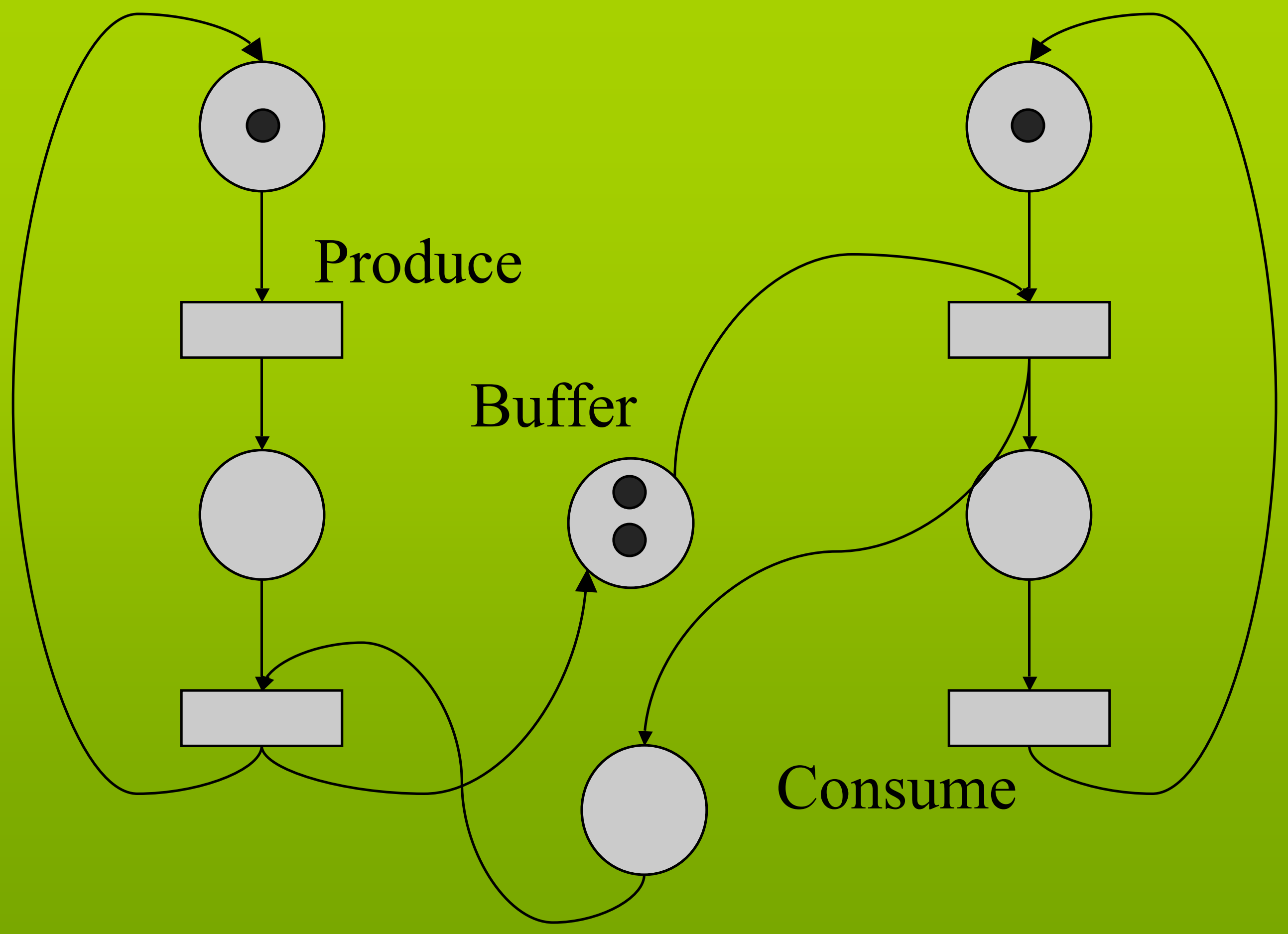

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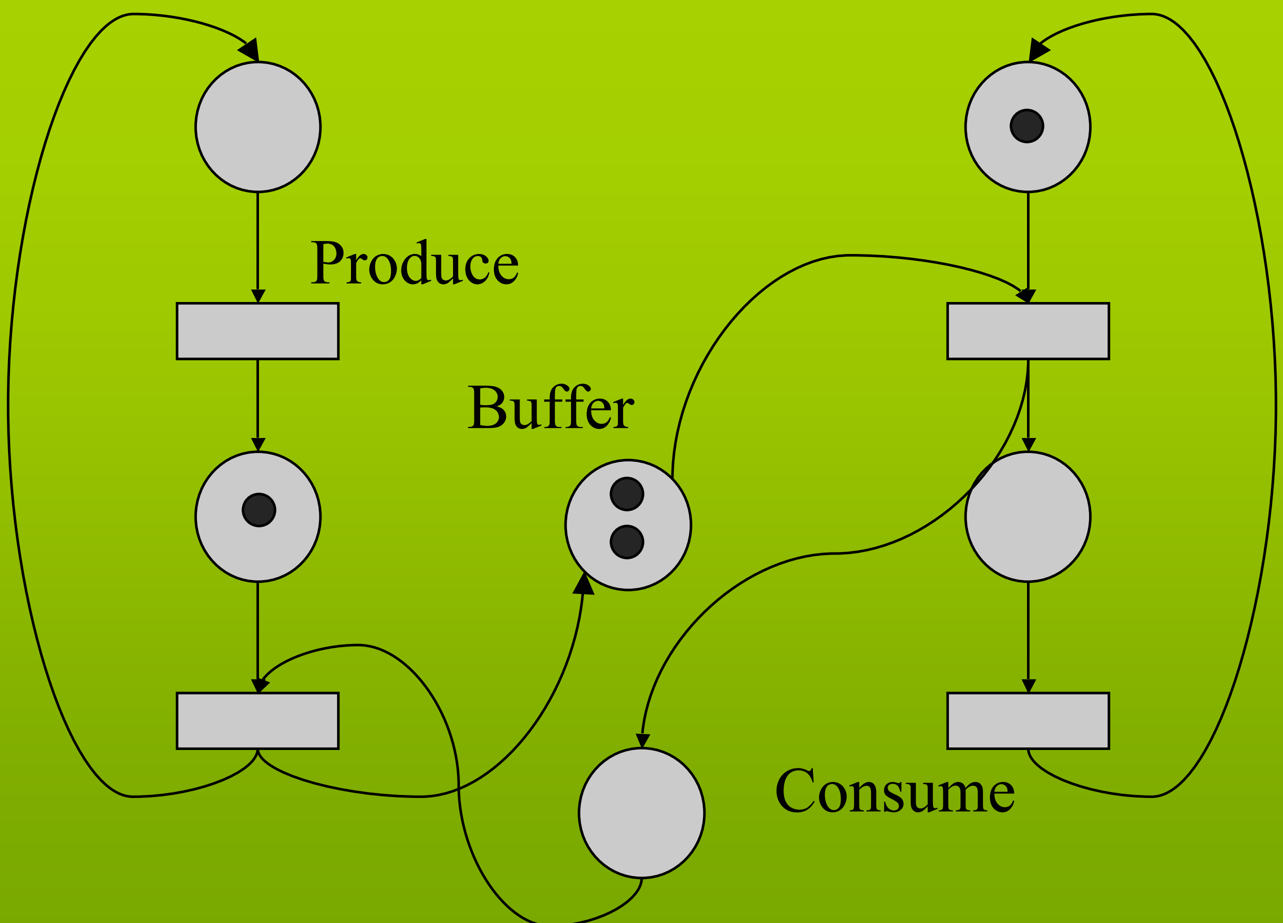
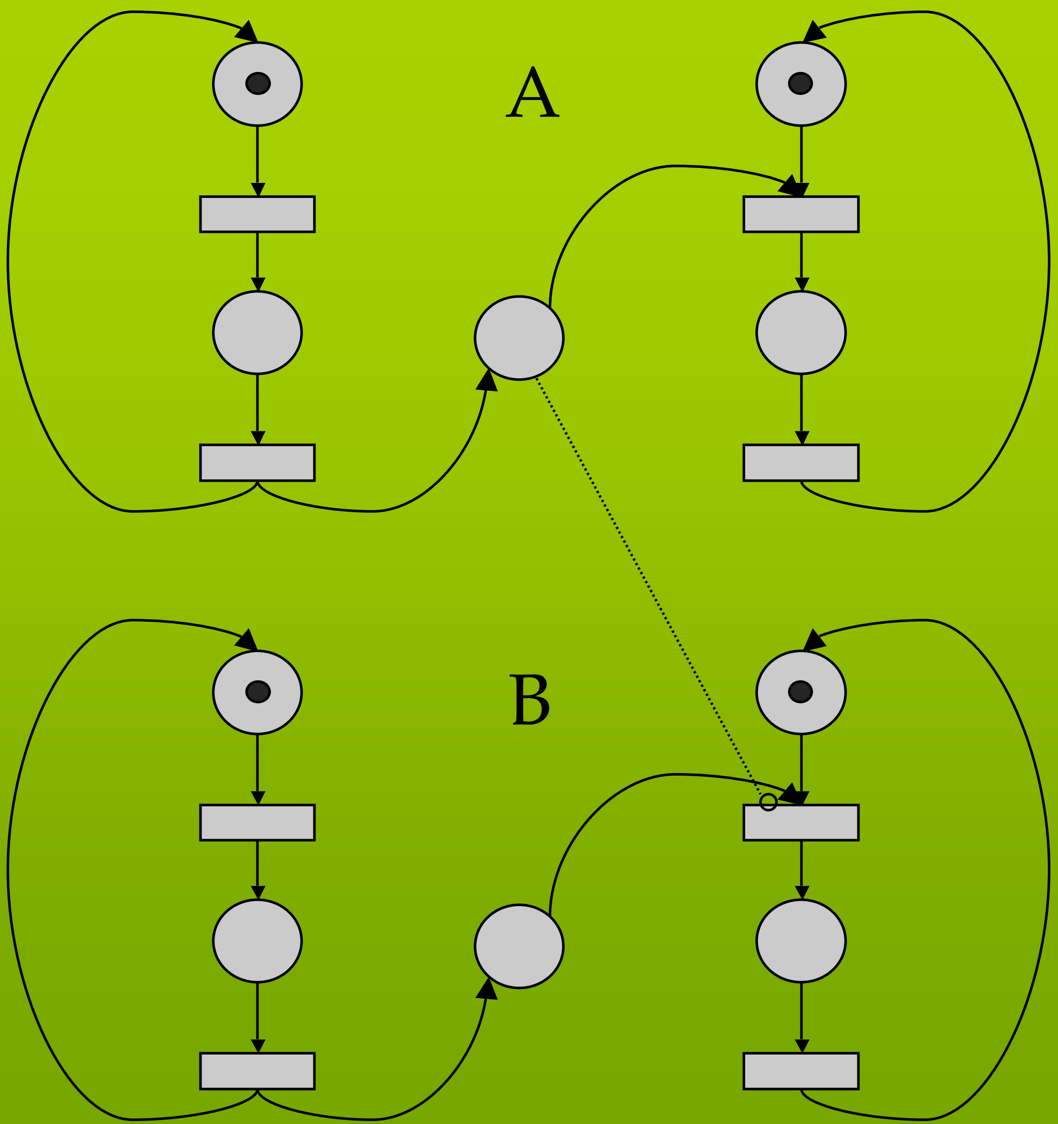


# Producer-Consumer with priority

**Consumer B can consume only if buffer A is empty**

**Inhibitor arcs**

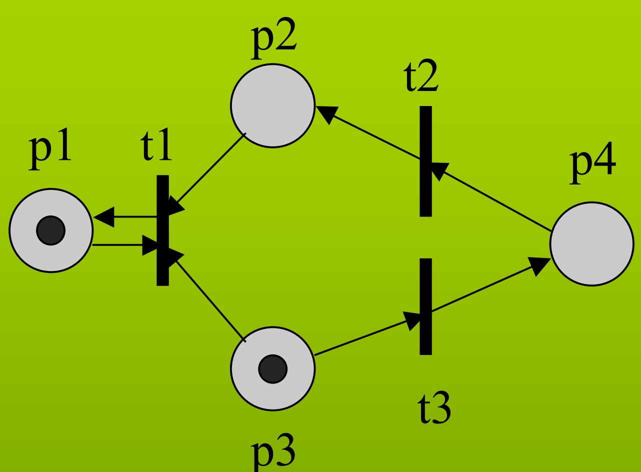


# PN properties

- **Behavioral: depend on the initial marking (most interesting)**
  - **Reachability**
  - **Boundedness**
  - **Schedulability**
  - **Liveness**
  - **Conservation**
- **Structural: do not depend on the initial marking (often too restrictive)**
  - **Consistency**
  - **Structural boundedness**

# Reachability

- **Marking M is reachable from marking $M_0$ if there exists a sequence of firings $\sigma = M_0\ t_1\ M_1\ t_2\ M_2 \ldots M$ that transforms $M_0$ to M.**
- **The reachability problem is decidable.**

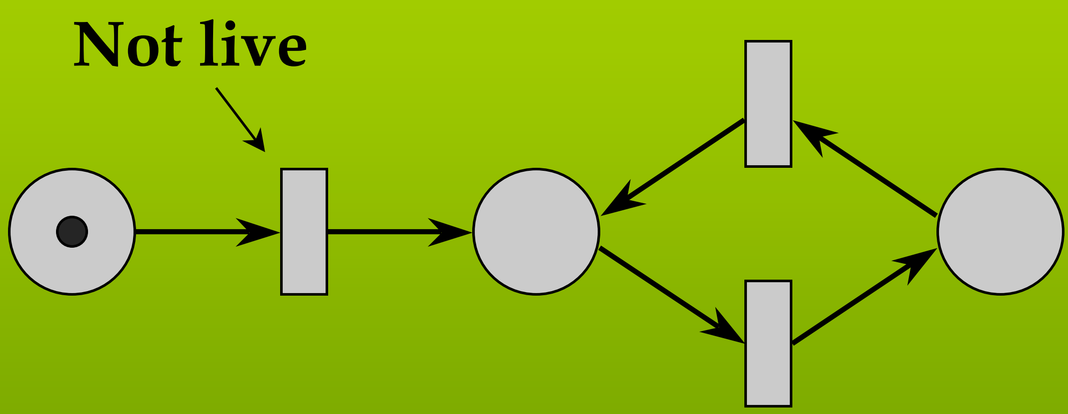



$M_0 = (1,0,1,0)$
$M = (1,1,0,0)$

$$M_0 = (1,0,1,0) \xrightarrow{t3} M_1 = (1,0,0,1) \xrightarrow{t2} M = (1,1,0,0)$$

# Liveness

- **Liveness**: from any marking any transition can become fireable
  - Liveness implies deadlock freedom, not viceversa



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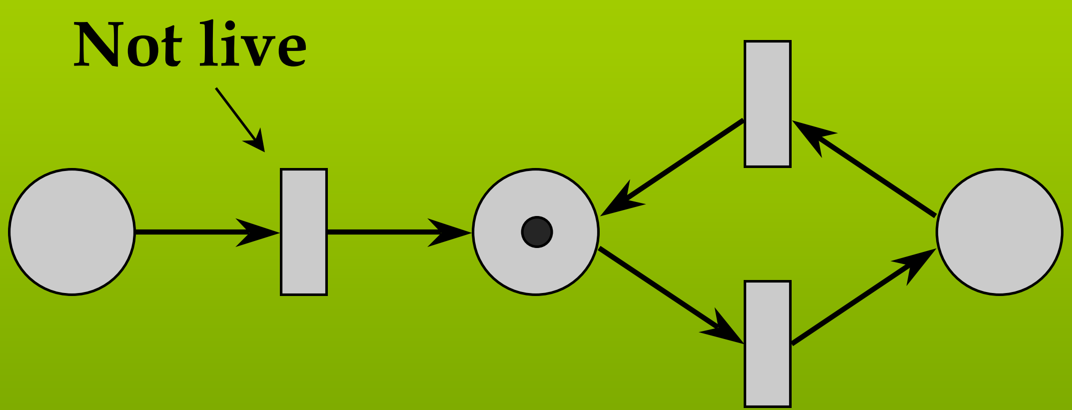

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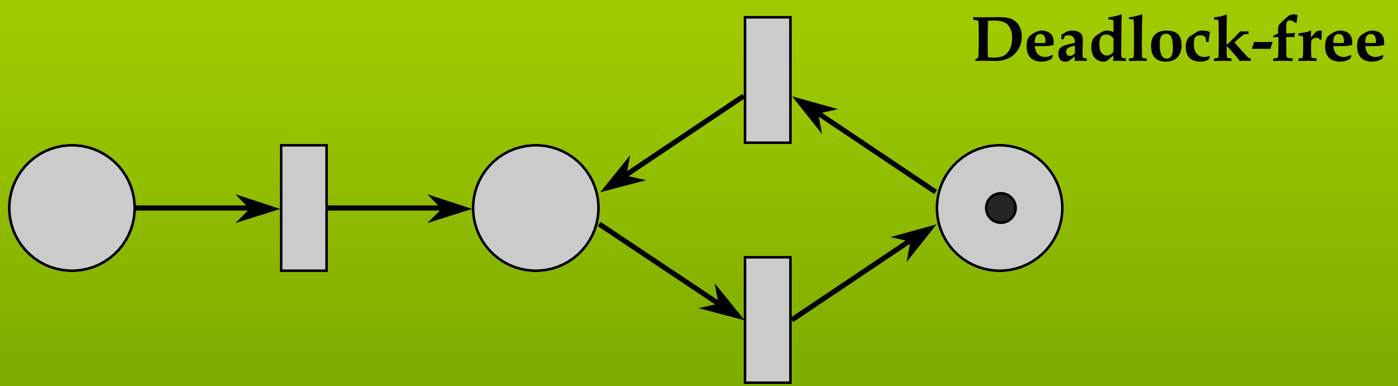

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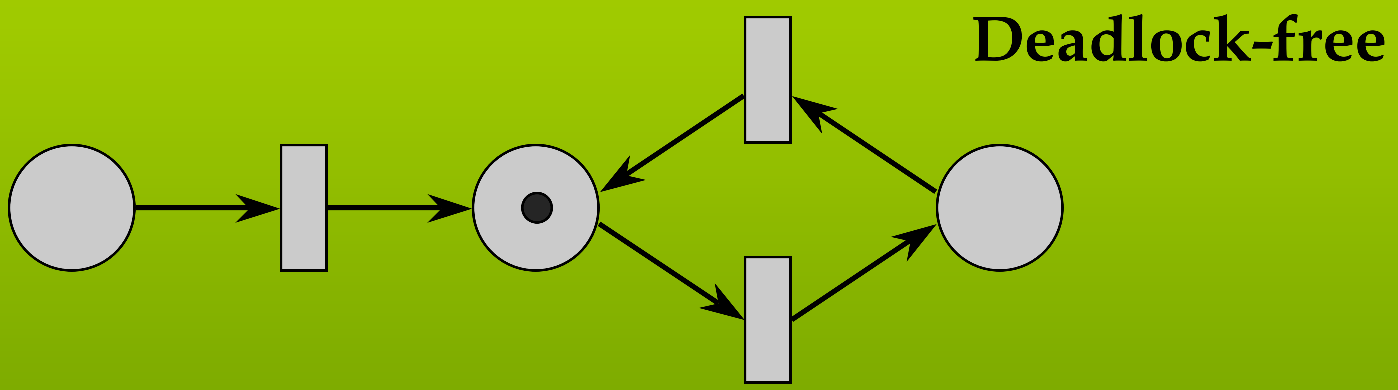
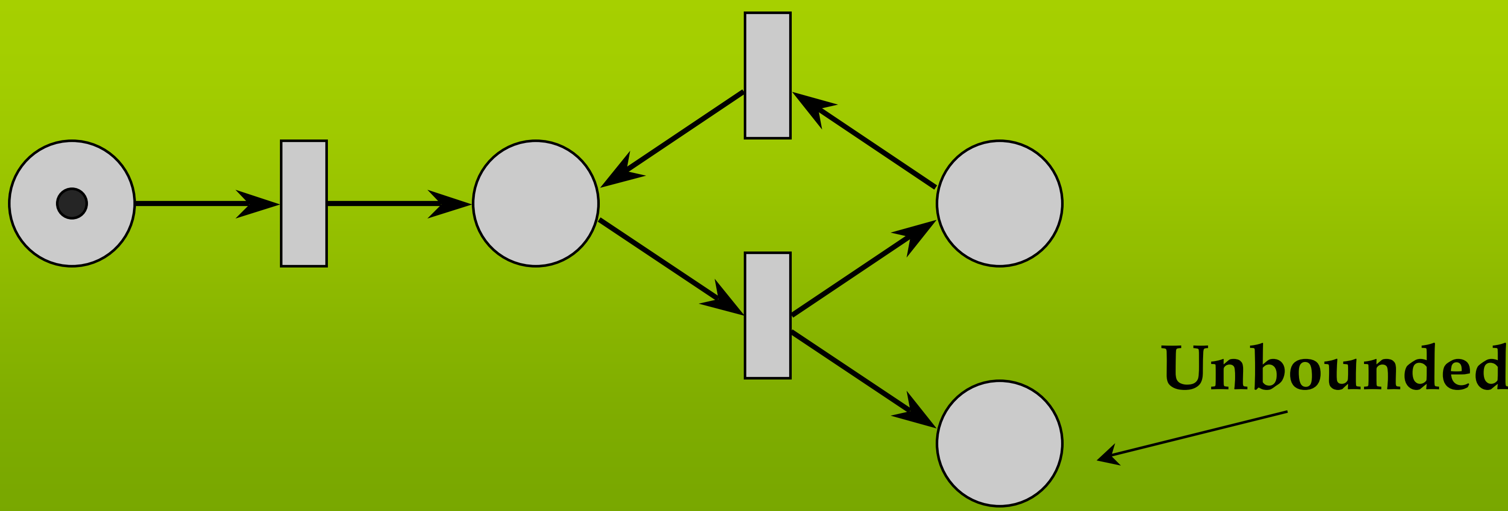


# Boundedness

- **Boundedness: the number of tokens in any place cannot grow indefinitely**
  - **(1-bounded also called *safe*)**
  - **Application: places represent buffers and registers (check there is no overflow)**



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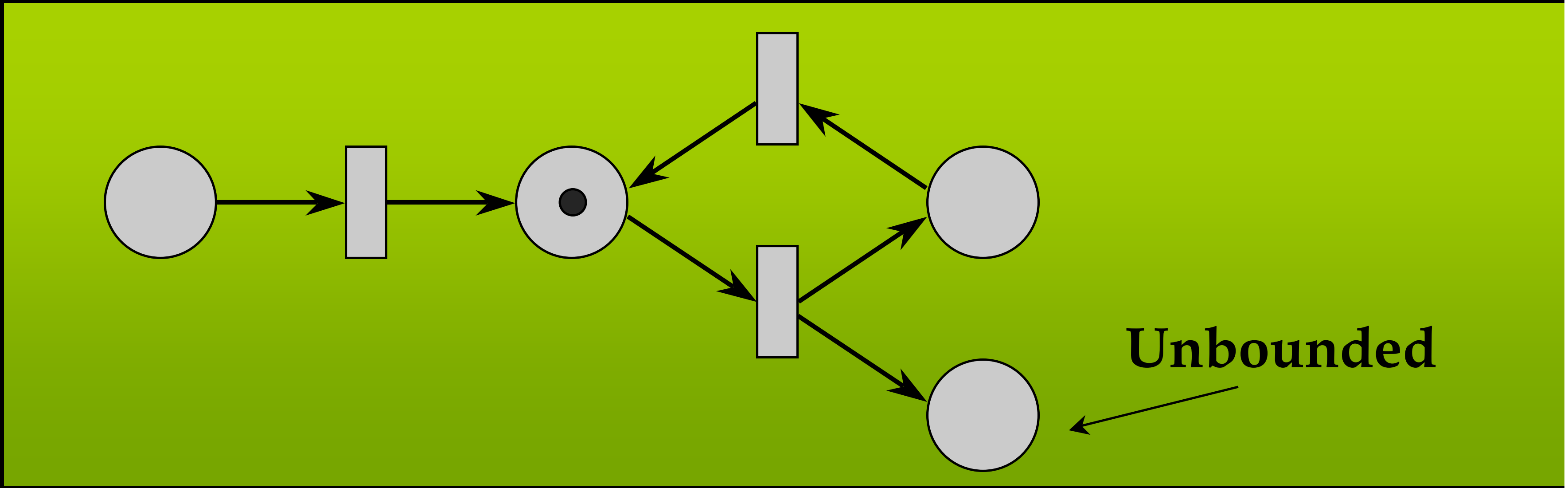

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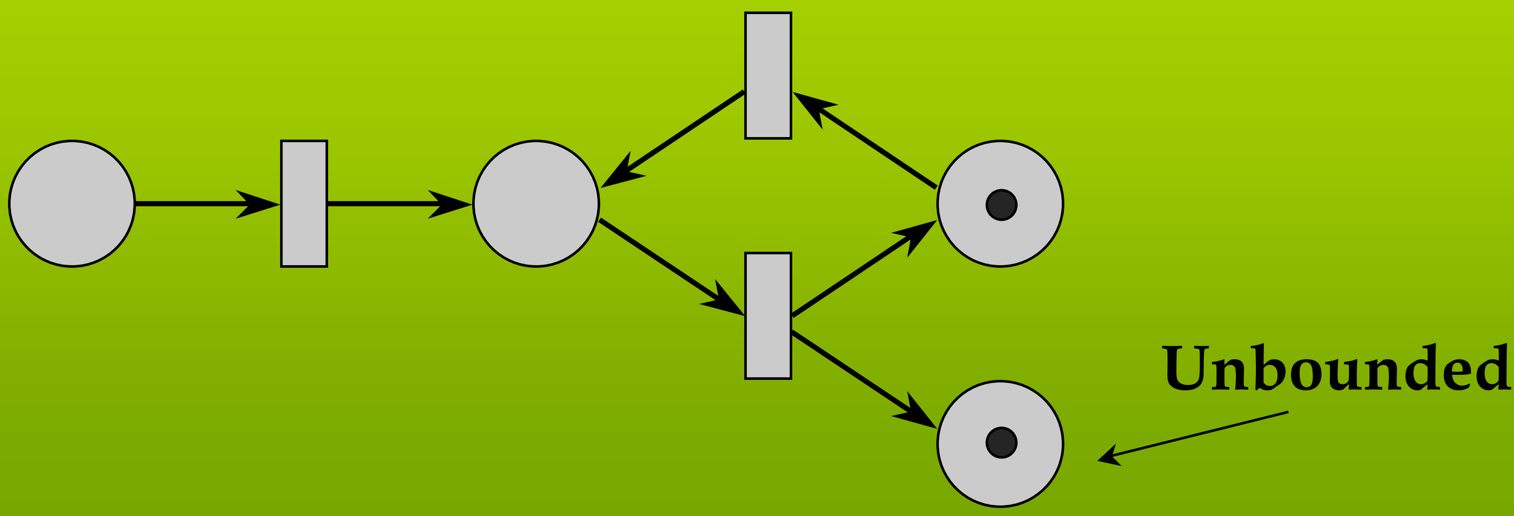

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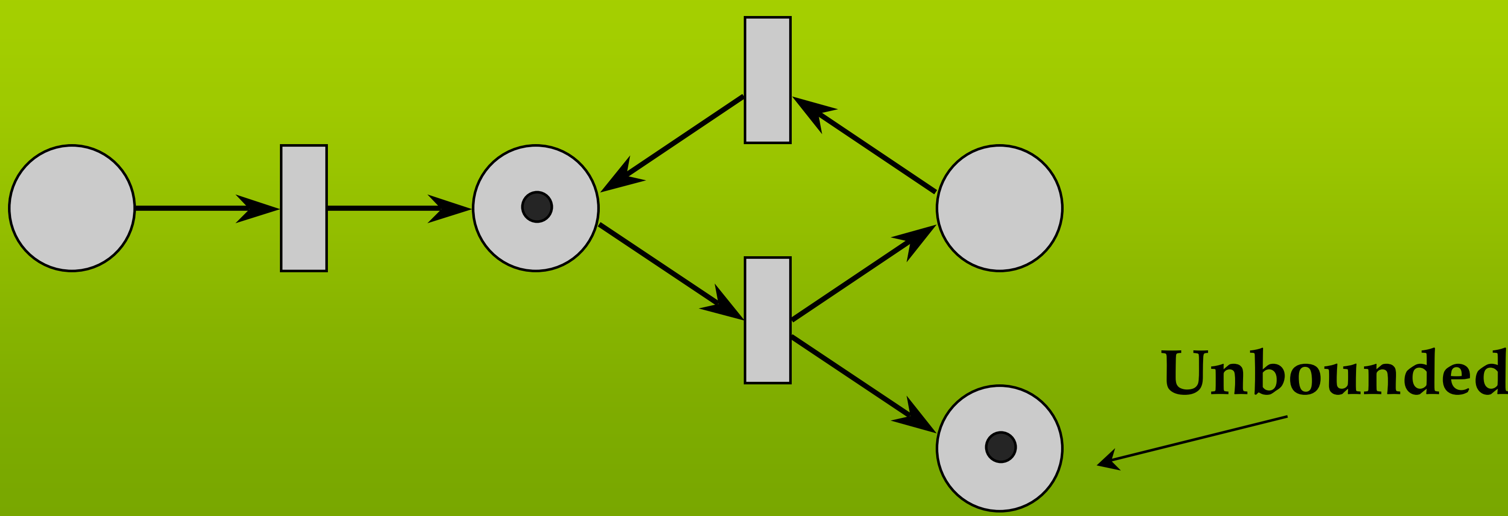

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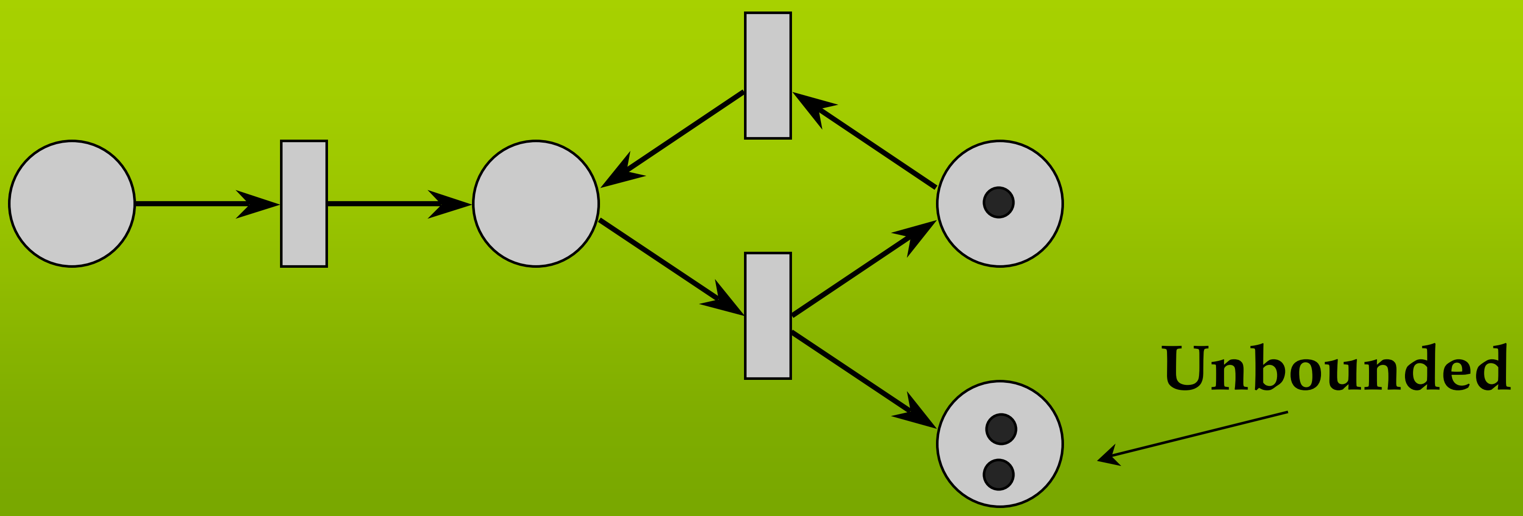
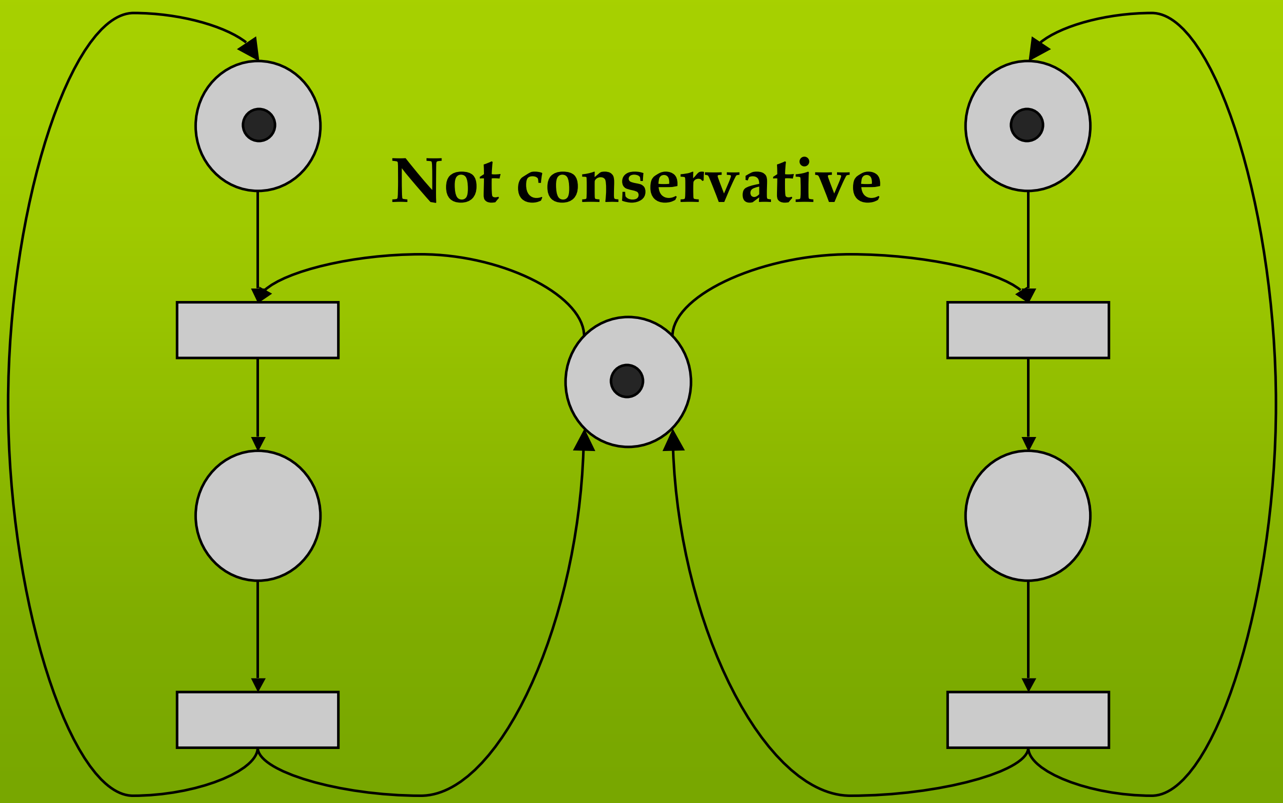


# Conservation

- **Conservation**: **the total number of tokens in the net is constant**



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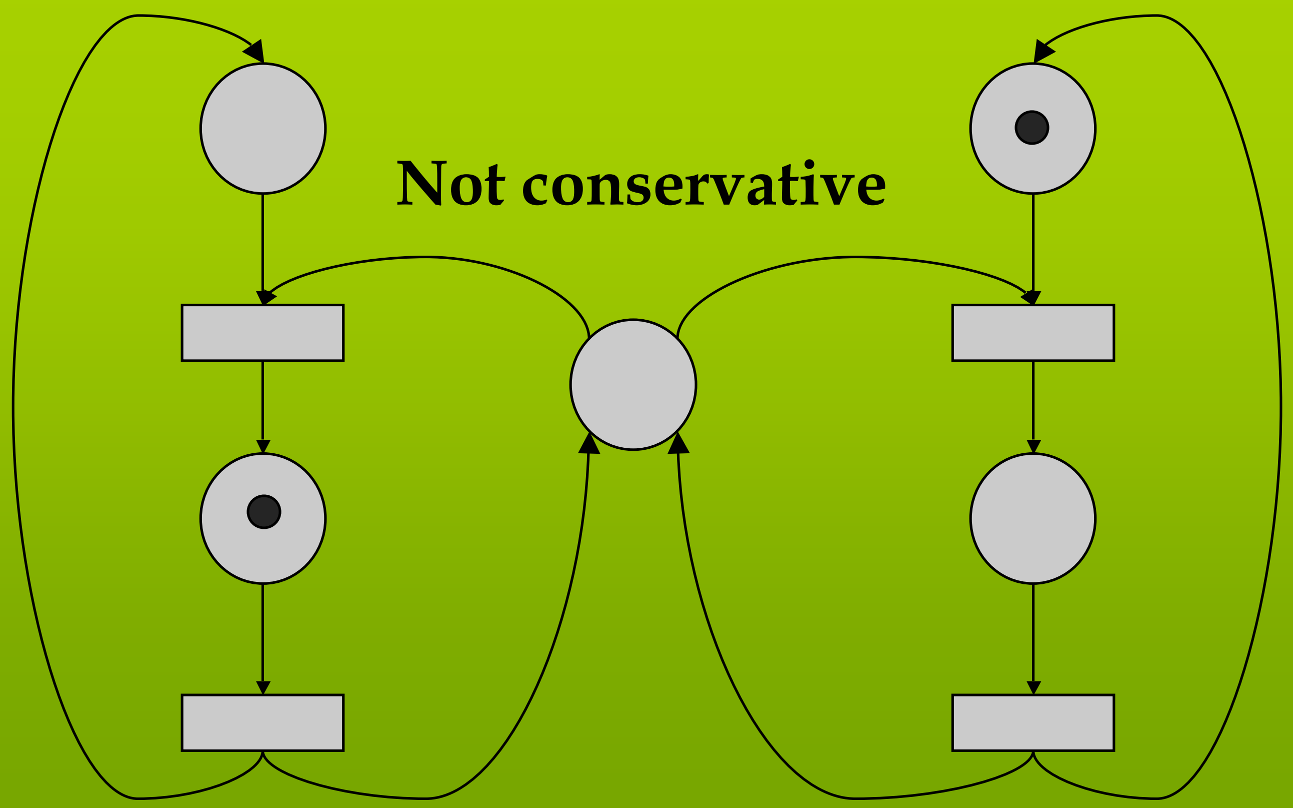

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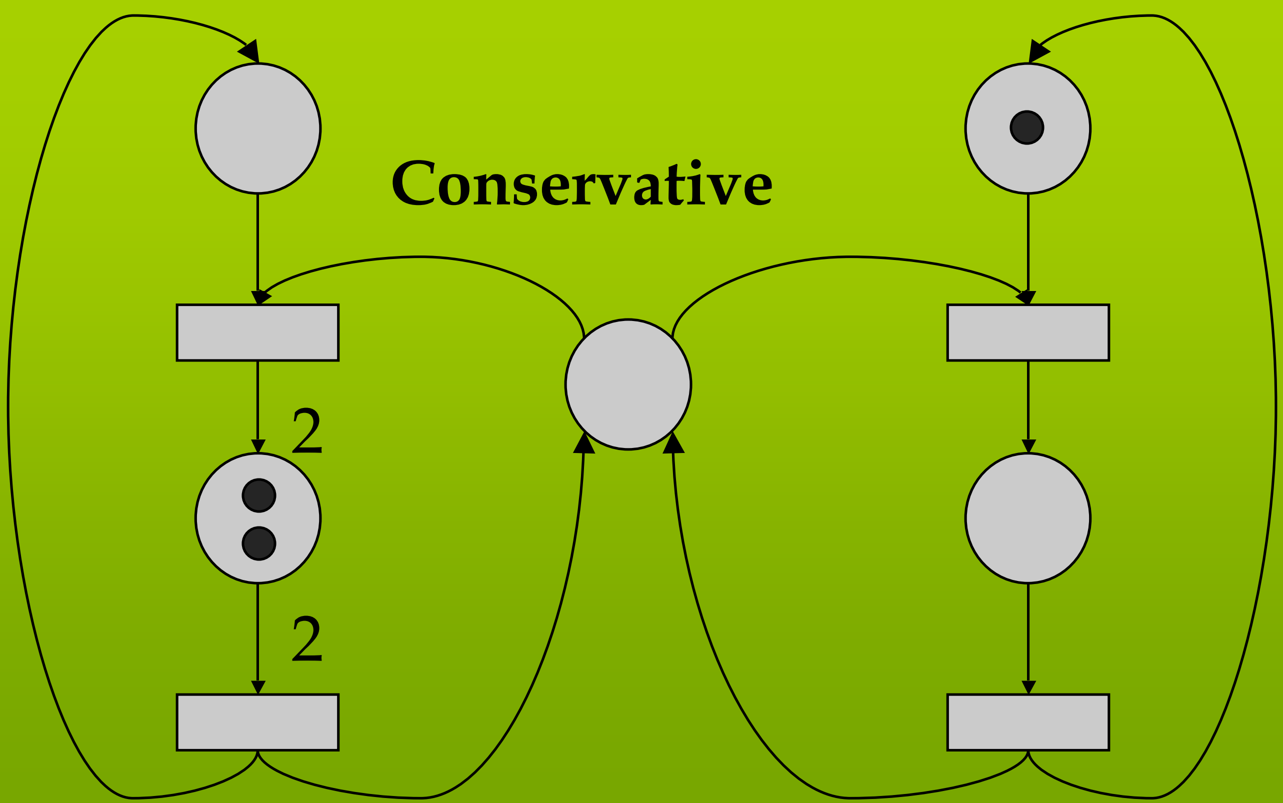
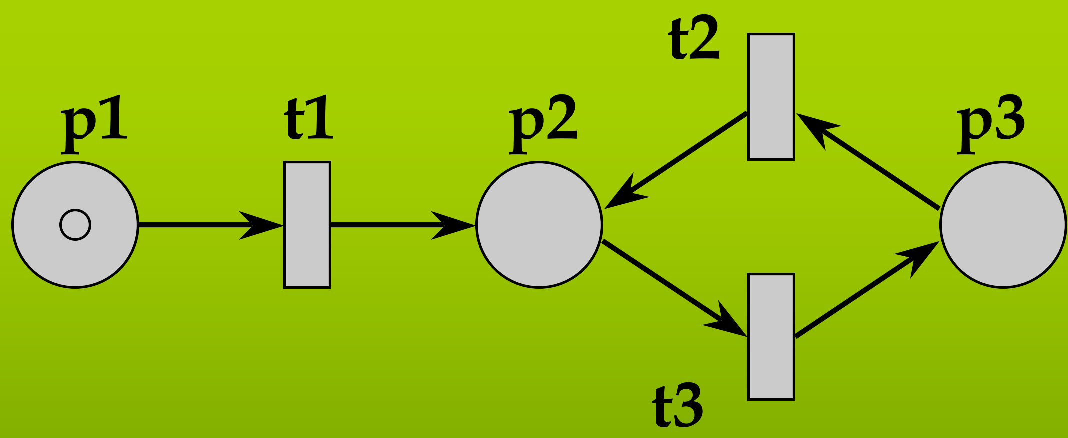


# Analysis techniques

- **Structural analysis techniques**
  - **Incidence matrix**
  - **T- and S- Invariants**
- **State Space Analysis techniques**
  - **Coverability Tree**
  - **Reachability Graph**

# Incidence Matrix




$$
A = \begin{array}{c c}
\begin{array}{ccc} t1 & t2 & t3 \end{array} & \\
\left|\begin{array}{ccc} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right| & \begin{array}{c} p1 \\ p2 \\ p3 \end{array}
\end{array}
$$

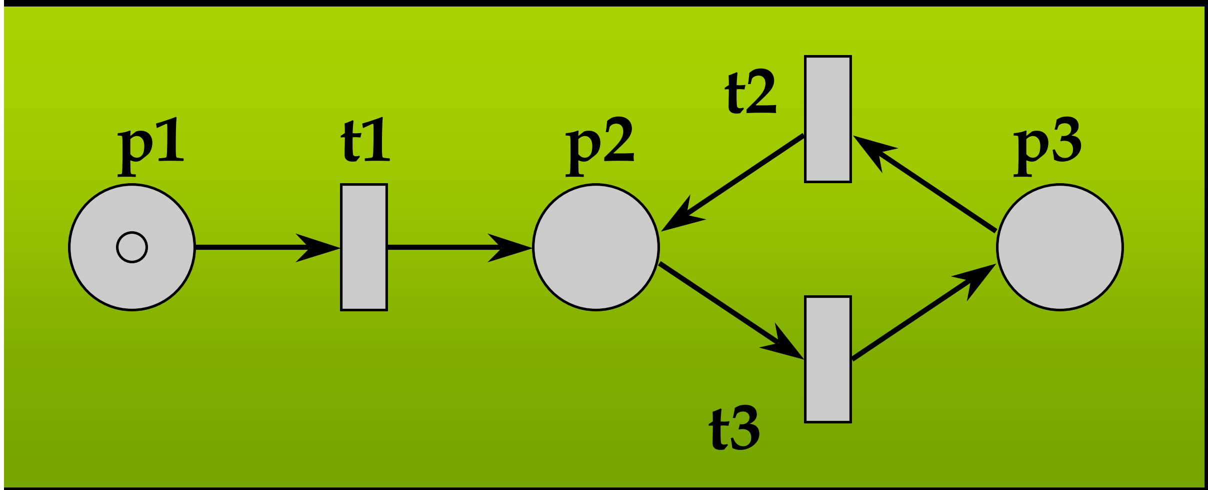
- **Necessary condition for marking M to be reachable from initial marking $M_0$:**

**there exists firing vector v s.t.:**

$$M = M_0 + A\,v$$

# State equations

- **E.g. reachability of M = $|0\ 0\ 1|^T$ from $M_0 = |1\ 0\ 0|^T$**

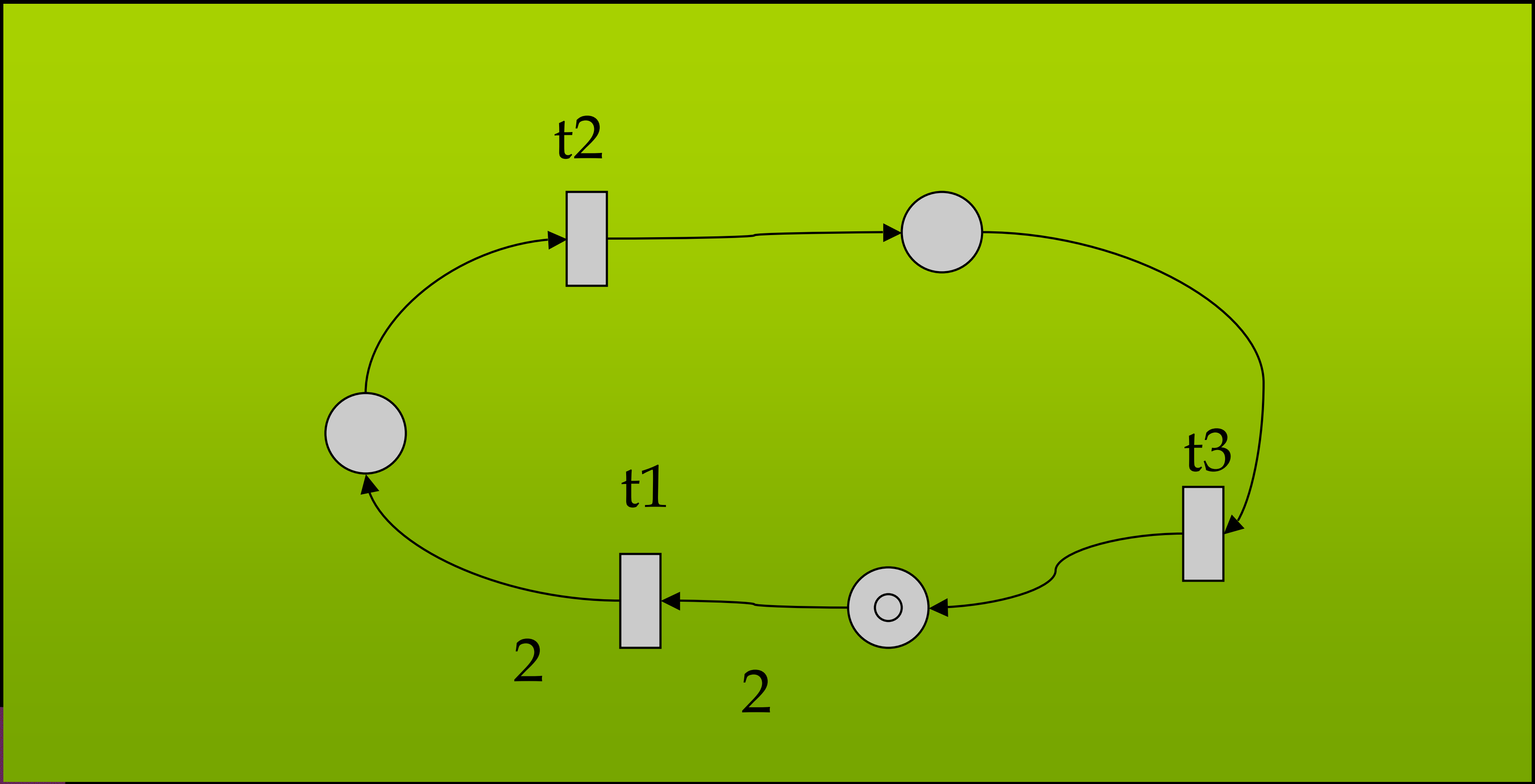



$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$v_1 = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \qquad \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

but also $v_2 = |\ 1\ 1\ 2\ |^T$ or any $v_k = |\ 1\ (k)\ (k+1)\ |^T$

# Necessary Condition only



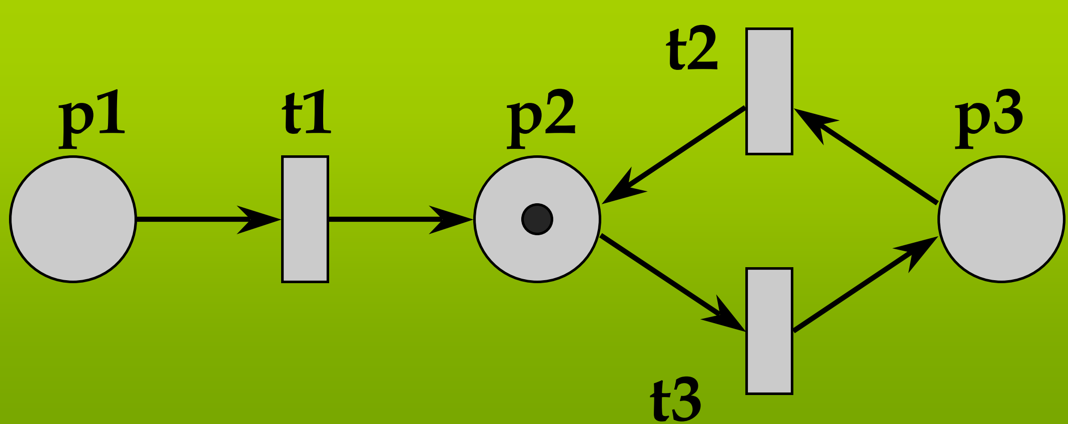

**Deadlock!!**

# State equations and invariants

- **Solutions of Ax = 0 (in $M = M_0 + Ax$, $M = M_0$)**

  **T-invariants**

  - **sequences of transitions that (if fireable) bring back to original marking**
  - **periodic schedule in SDF**
  - **e.g. $x = | 0\ 1\ 1 |^T$**




$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

# Application of T-invariants

- **Scheduling**
  - **Cyclic schedules: need to return to the initial state**

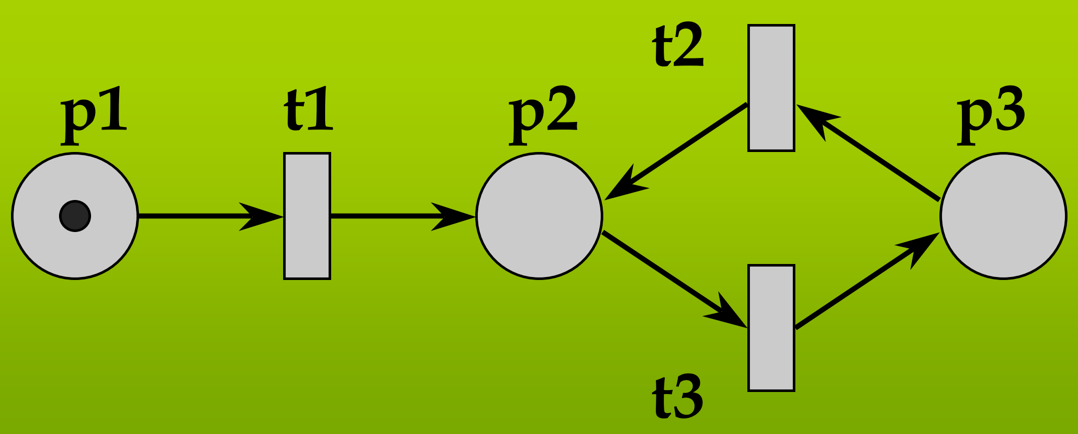
# State equations and invariants

- **Solutions of yA = 0**

  **S-invariants**

  – **sets of places whose weighted total token count does not change after the firing of any transition (y M = y M')**

  – **e.g. $y = | 1\ 1\ 1 |^T$**




$$A^T = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

# Application of S-invariants

- **Structural Boundedness: bounded for any finite initial marking $M_0$**
- **Existence of a positive S-invariant is CS for structural boundedness**
  - **initial marking is finite**
  - **weighted token count does not change**

# Summary of algebraic methods

- **Extremely efficient**

  **(polynomial in the size of the net)**
- **Generally provide only necessary or sufficient information**
- **Excellent for ruling out some deadlocks or otherwise dangerous conditions**
- **Can be used to infer structural boundedness**

# Coverability Tree

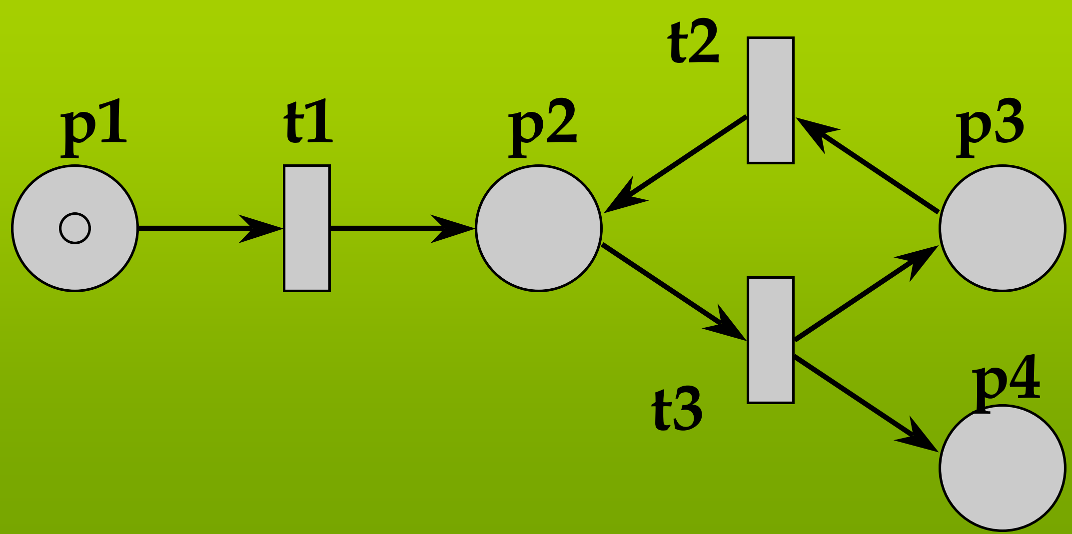
- **Build a (finite) tree representation of the markings**

**Karp-Miller algorithm**

- **Label initial marking M0 as the root of the tree and tag it as *new***
- **While new markings exist do:**
  - **select a new marking M**
  - **if M is identical to a marking on the path from the root to M, then tag M as *old* and go to another new marking**
  - **if no transitions are enabled at M, tag M *dead-end***
  - **while there exist enabled transitions at M do:**
    - **obtain the marking M' that results from firing t at M**
    - **on the path from the root to M if there exists a marking M'' such that M'(p)>=M''(p) for each place p and M' is different from M'', then replace M'(p) by $\omega$ for each p such that M'(p) >M''(p)**
    - **introduce M' as a node, draw an arc with label t from M to M' and tag M' as *new*.**

# Coverability Tree

- **Boundedness is decidable with *coverability tree***



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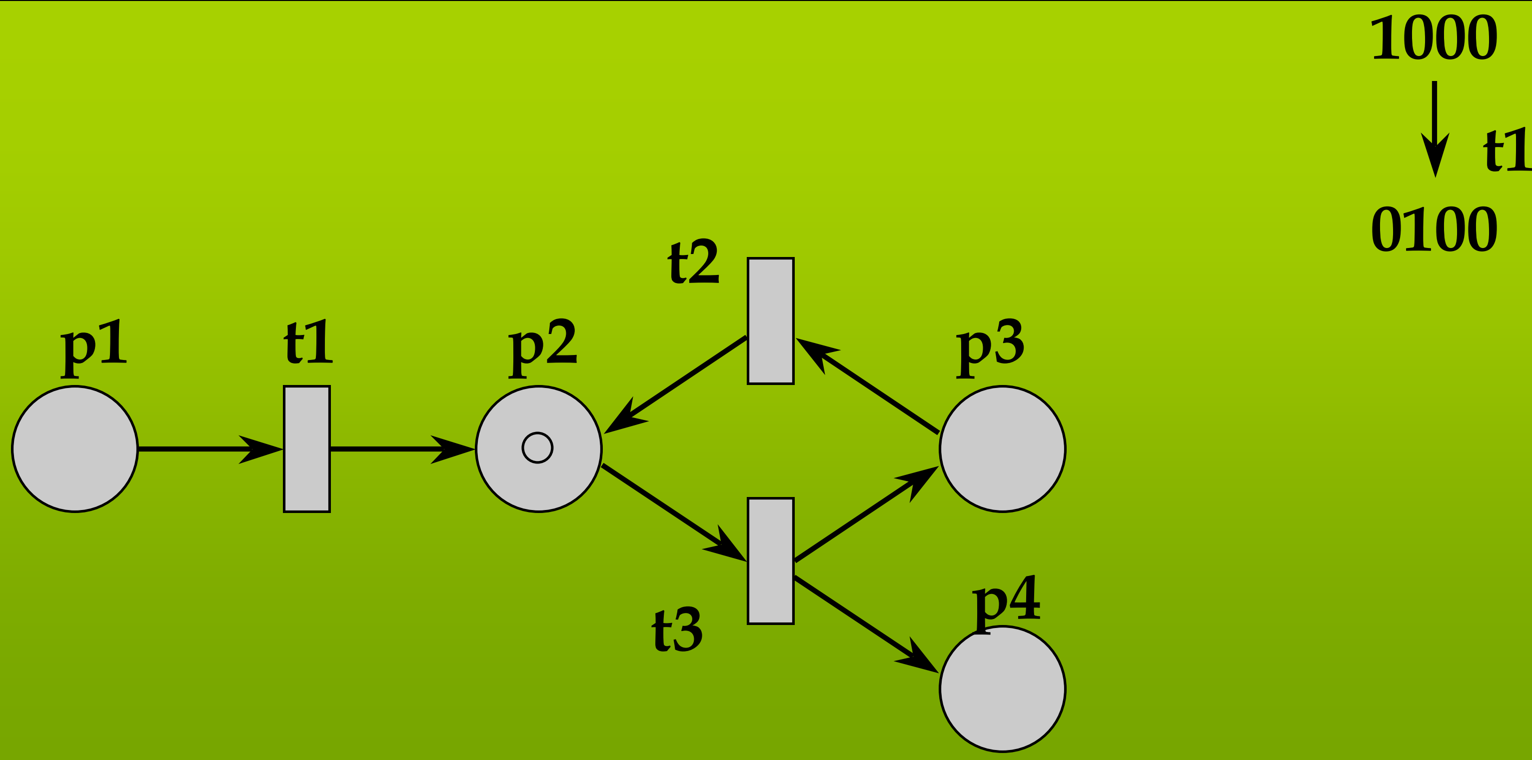

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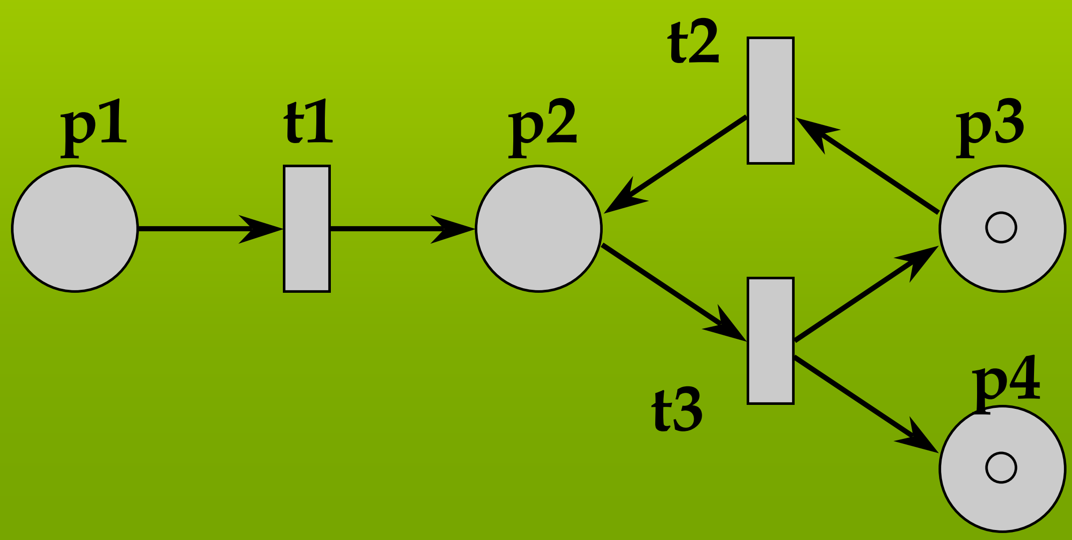


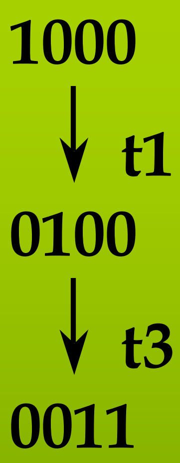

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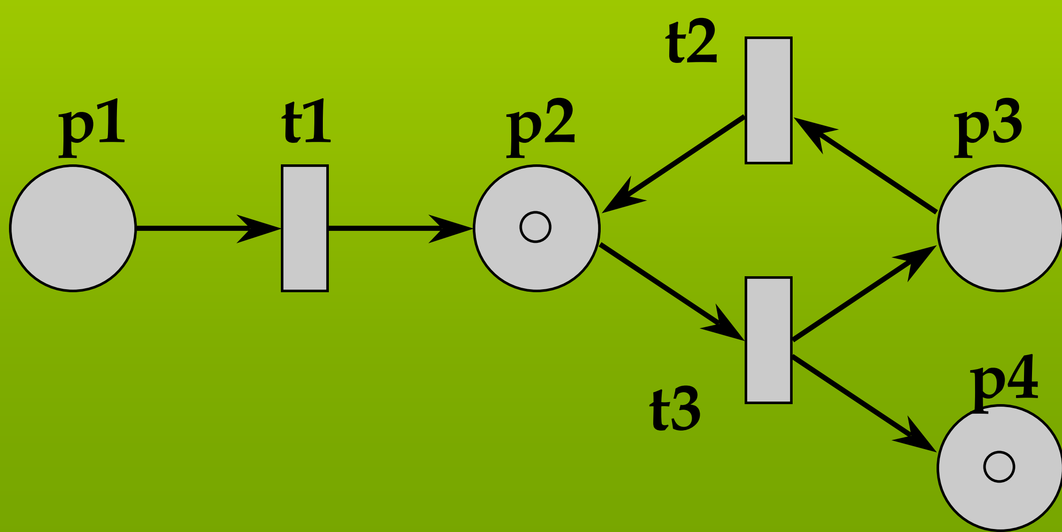


$$
\begin{array}{l}
1000 \\
\downarrow t1 \\
0100 \\
\downarrow t3 \\
0011 \\
\downarrow t2 \\
0101
\end{array}
$$

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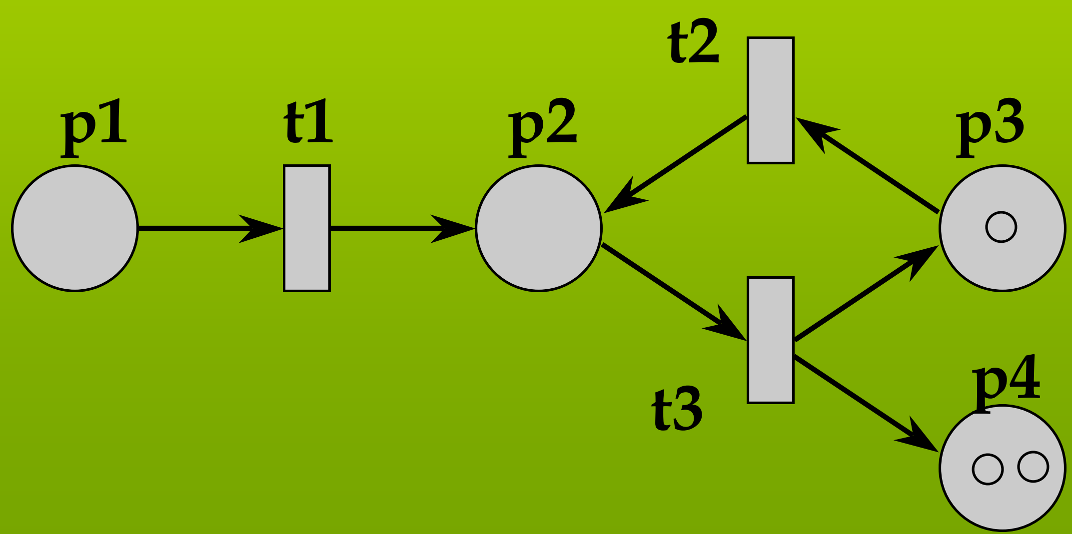


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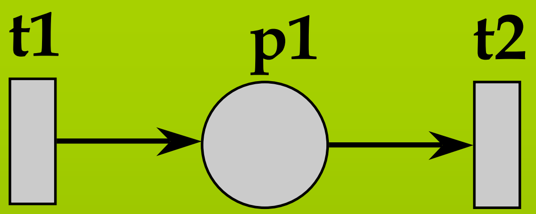
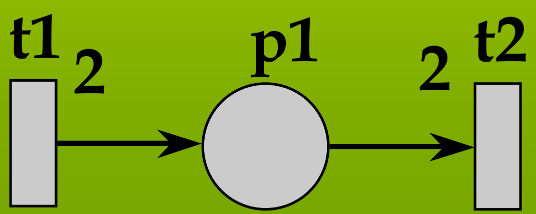
# Coverability Tree

- **Is (1) reachable from (0)?**



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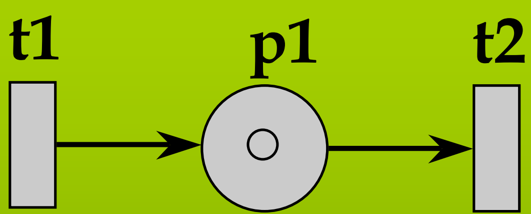
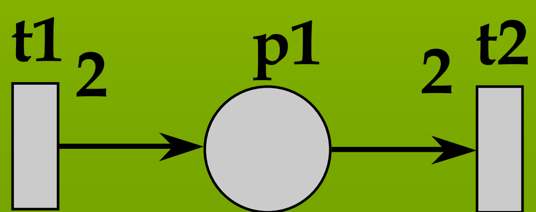



$$0 \xrightarrow{t1} \omega$$

$$(0) \longrightarrow (1) \longrightarrow (2) \ldots$$



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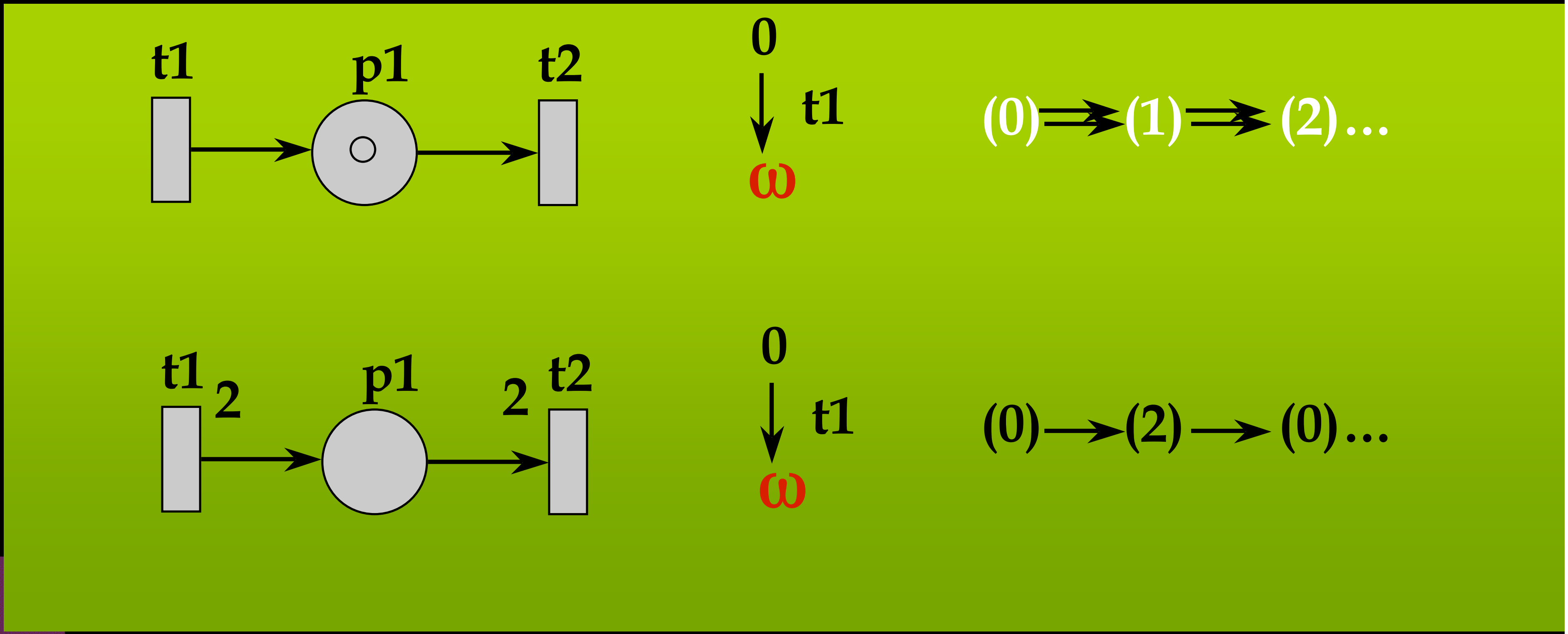

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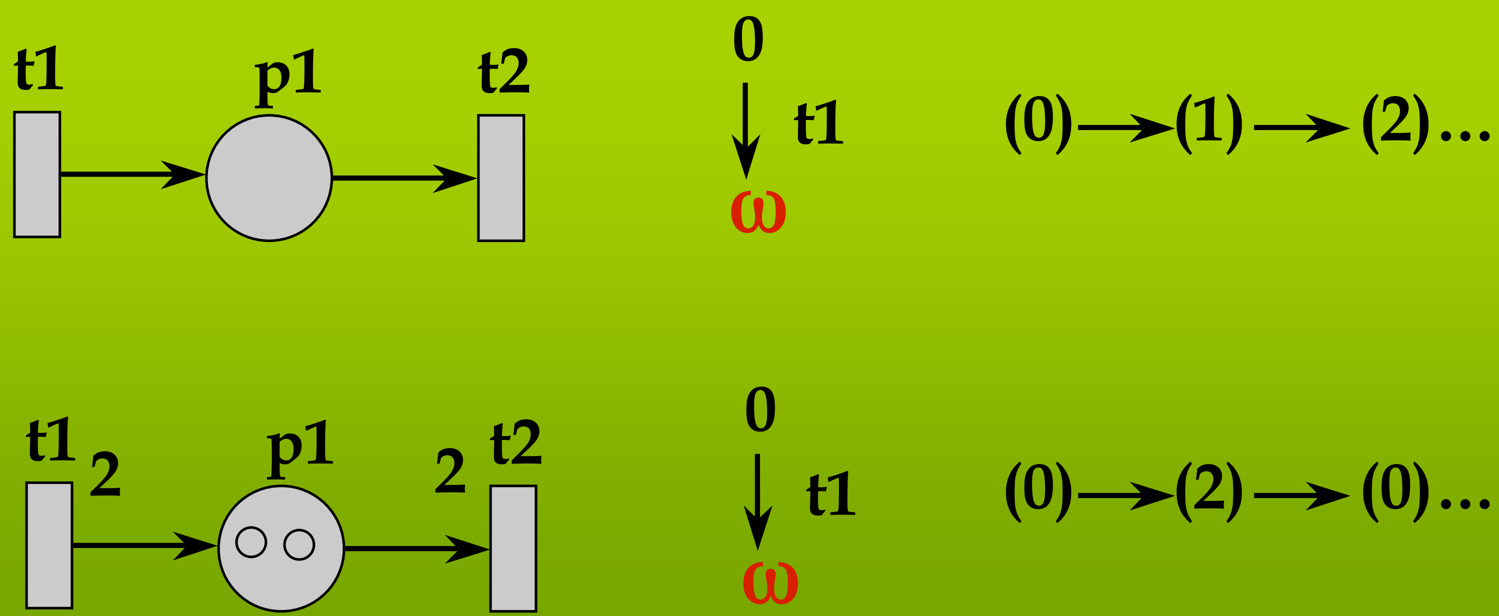
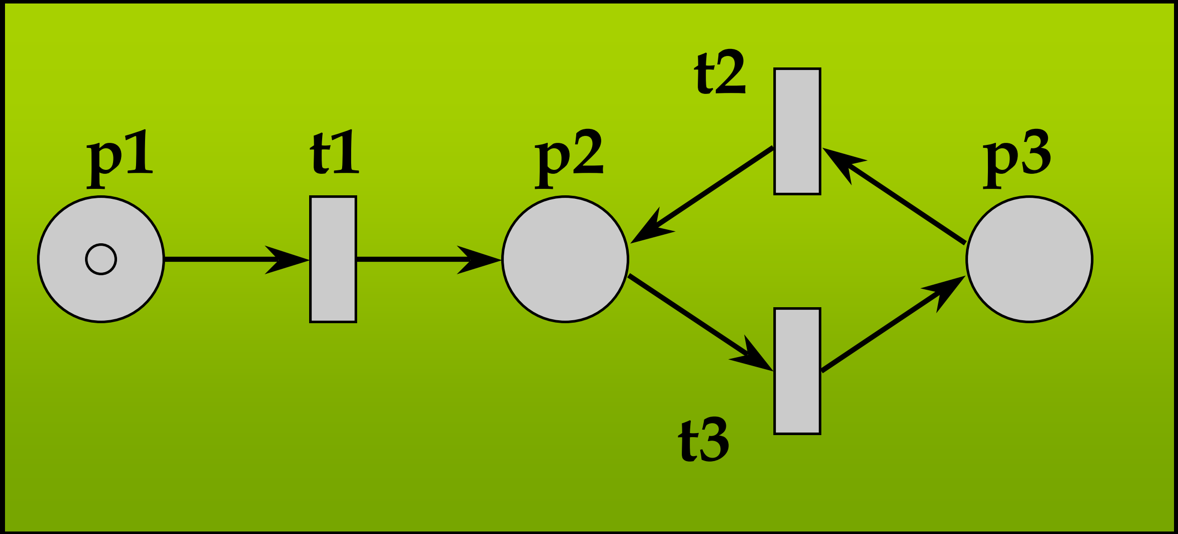


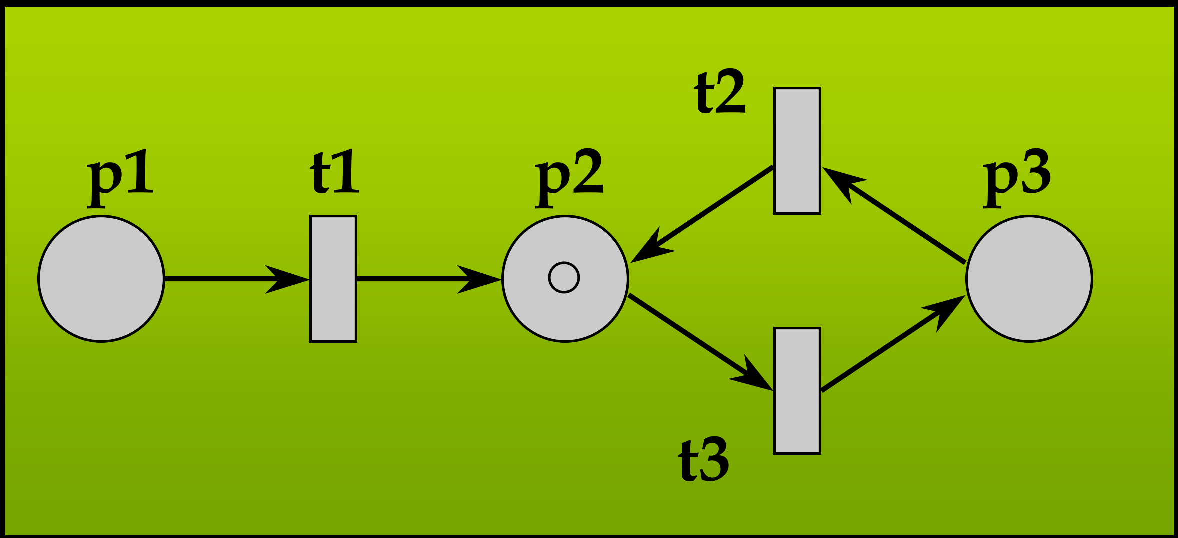

- **Cannot solve the reachability problem**

# Reachability graph




- **For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings**

# Reachability graph

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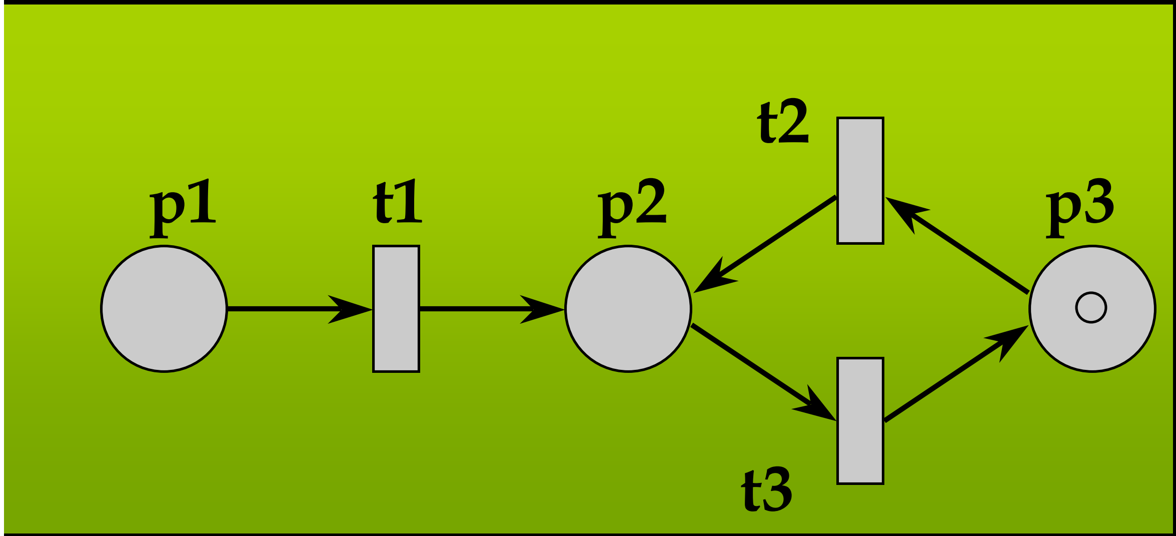


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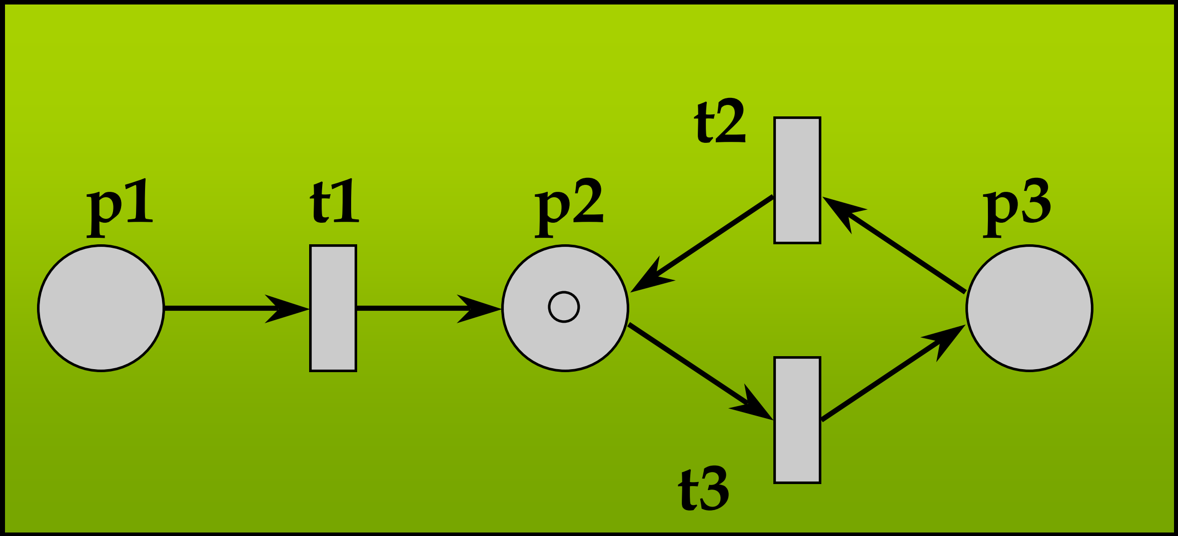
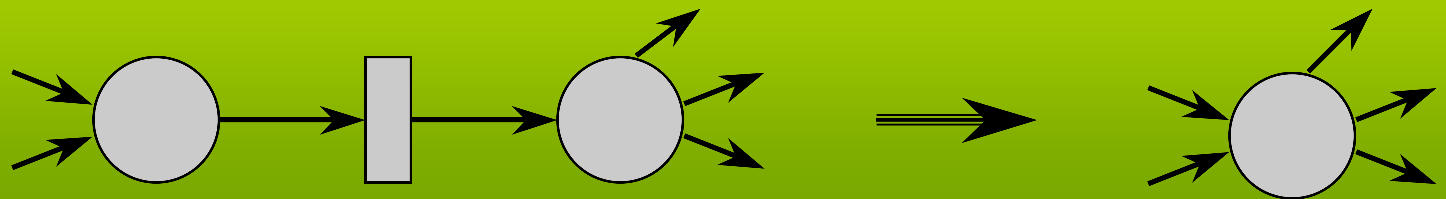



- For bounded nets the Coverability Tree is called Reachability Tree since it contains all possible reachable markings

# Subclasses of Petri nets

- Reachability analysis is too expensive
- State equations give only partial information
- Some properties are preserved by reduction rules

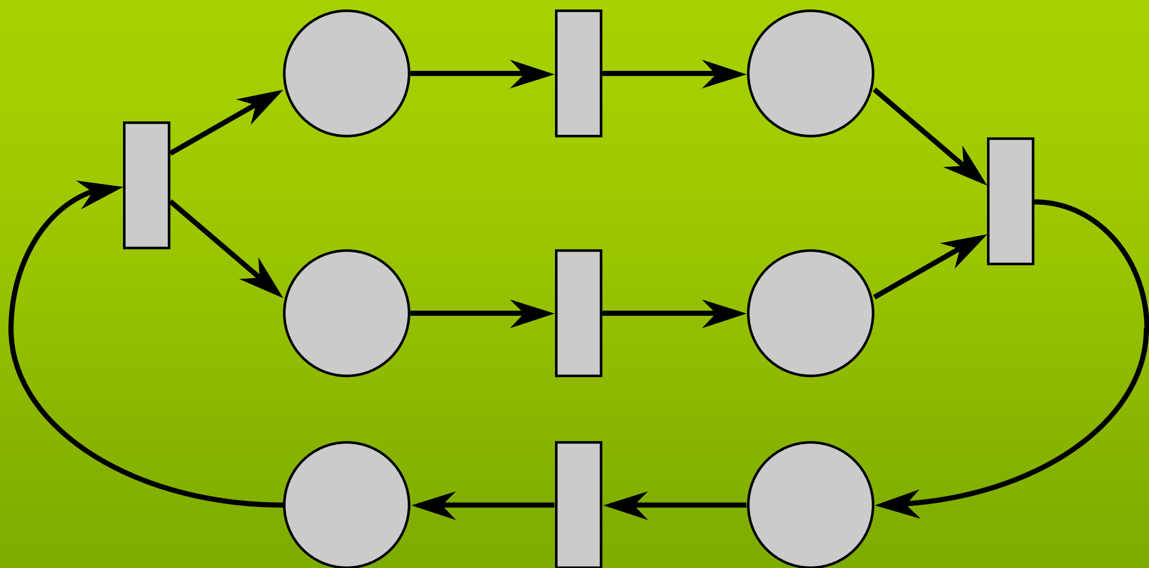
  e.g. for liveness and safeness



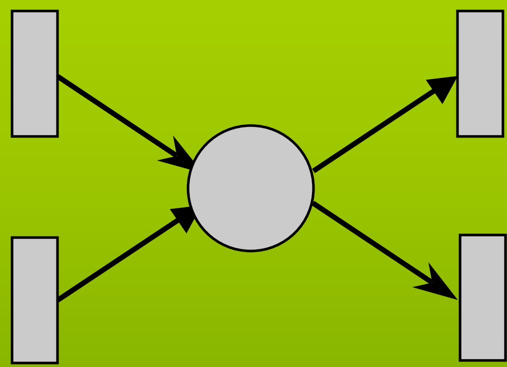
- Even reduction rules only work in some cases
- Must restrict class in order to prove stronger results

# Marked Graphs

- Every place has at most 1 predecessor and 1 successor transition
- Models only causality and concurrency (no conflict)



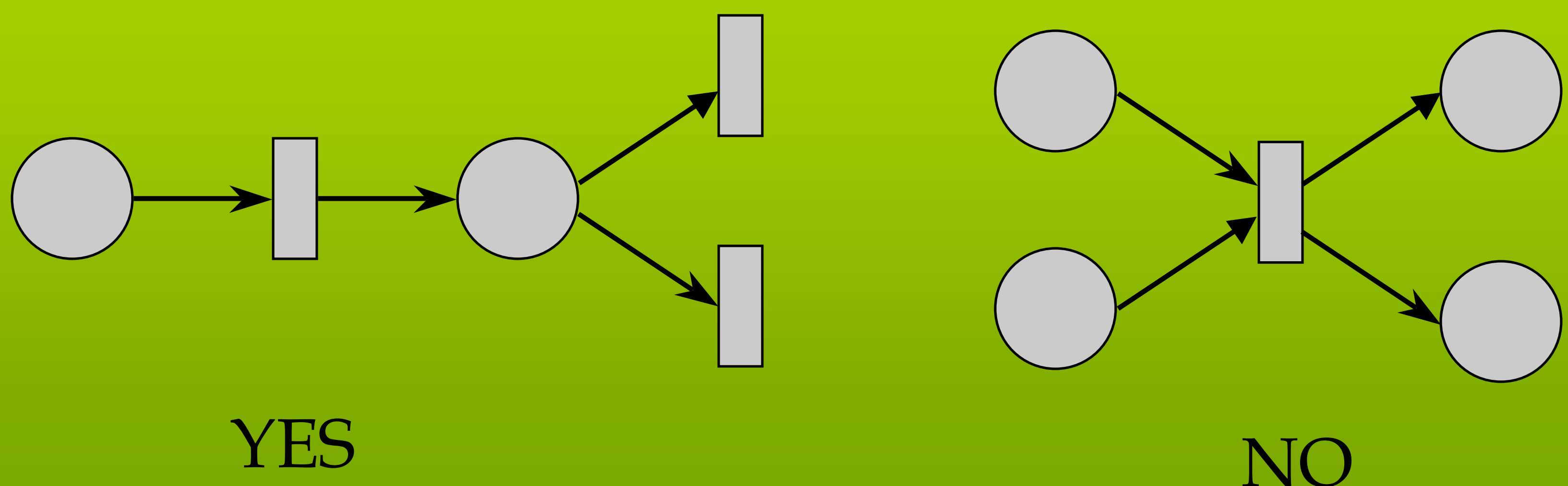
YES



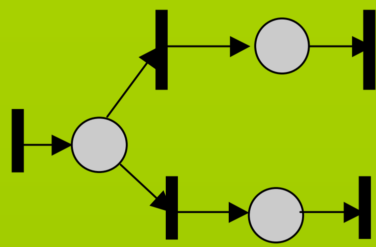
NO

# State Machines

- **Every transition has at most 1 predecessor and 1 successor place**
- **Models only causality and conflict**
  - **(no concurrency, no synchronization of parallel activities)**

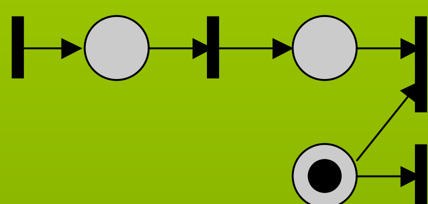


# Free-Choice Petri Nets (FCPN)

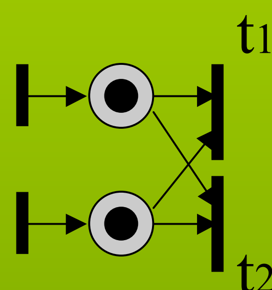


**every transition after choice has exactly 1 predecessor place**

**Free-Choice (FC)**



**Confusion (not-Free-Choice)**




**Extended Free-Choice**

**Free-Choice: the outcome of a choice depends on the value of a token (abstracted non-deterministically) rather than on its arrival time.**

# Free-Choice nets

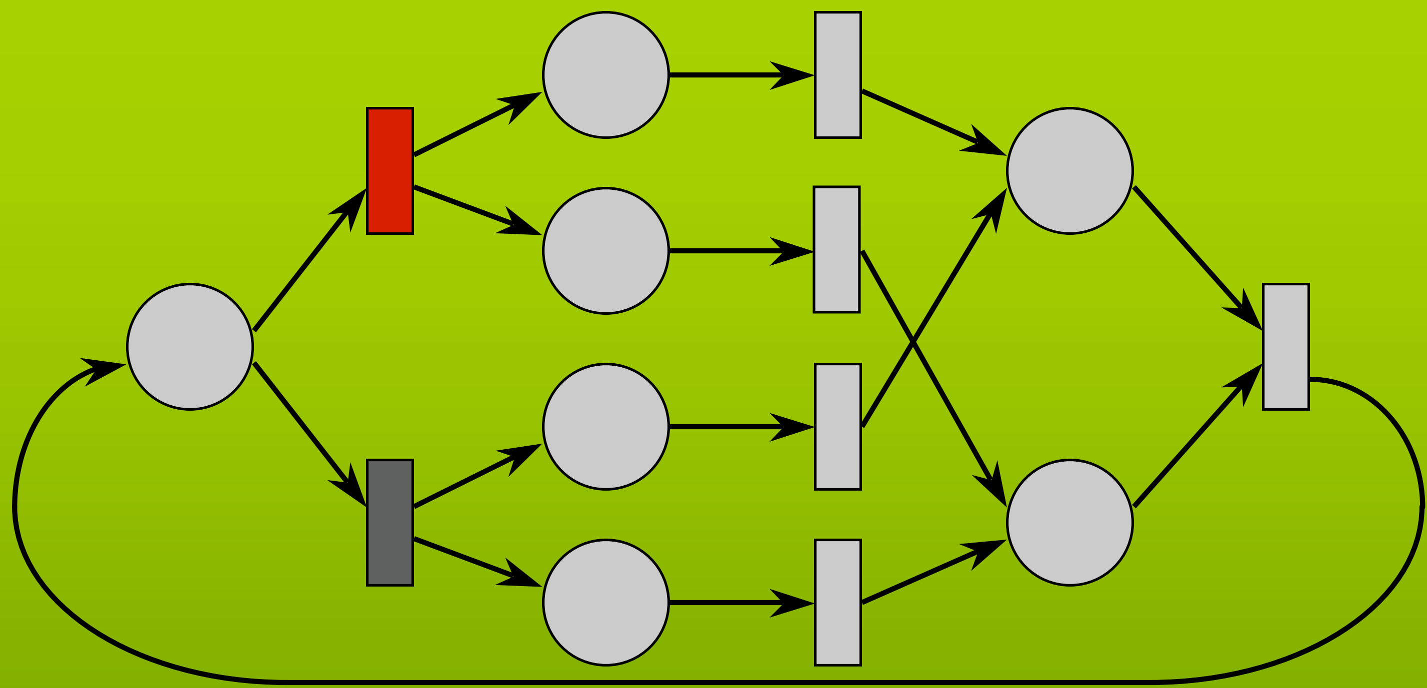
- **Introduced by Hack (‘72)**
- **Extensively studied by Best (‘86) and Desel and Esparza (‘95)**
- **Can express concurrency, causality and choice without confusion**
- **Very strong structural theory**
  - **necessary and sufficient conditions for liveness and safeness, based on decomposition**
  - **exploits duality between MG and SM**

# MG (& SM) decomposition

- An **Allocation** is a control function that chooses which transition fires among several conflicting ones ( A: P T).
- Eliminate the subnet that would be inactive if we were to use the allocation...
- **Reduction Algorithm**
  - Delete all unallocated transitions
  - Delete all places that have all input transitions already deleted
  - Delete all transitions that have at least one input place already deleted
- Obtain a **Reduction** (one for each allocation) that is a conflict free subnet

# MG reduction and cover

- **Choose one successor for each conflicting place:**



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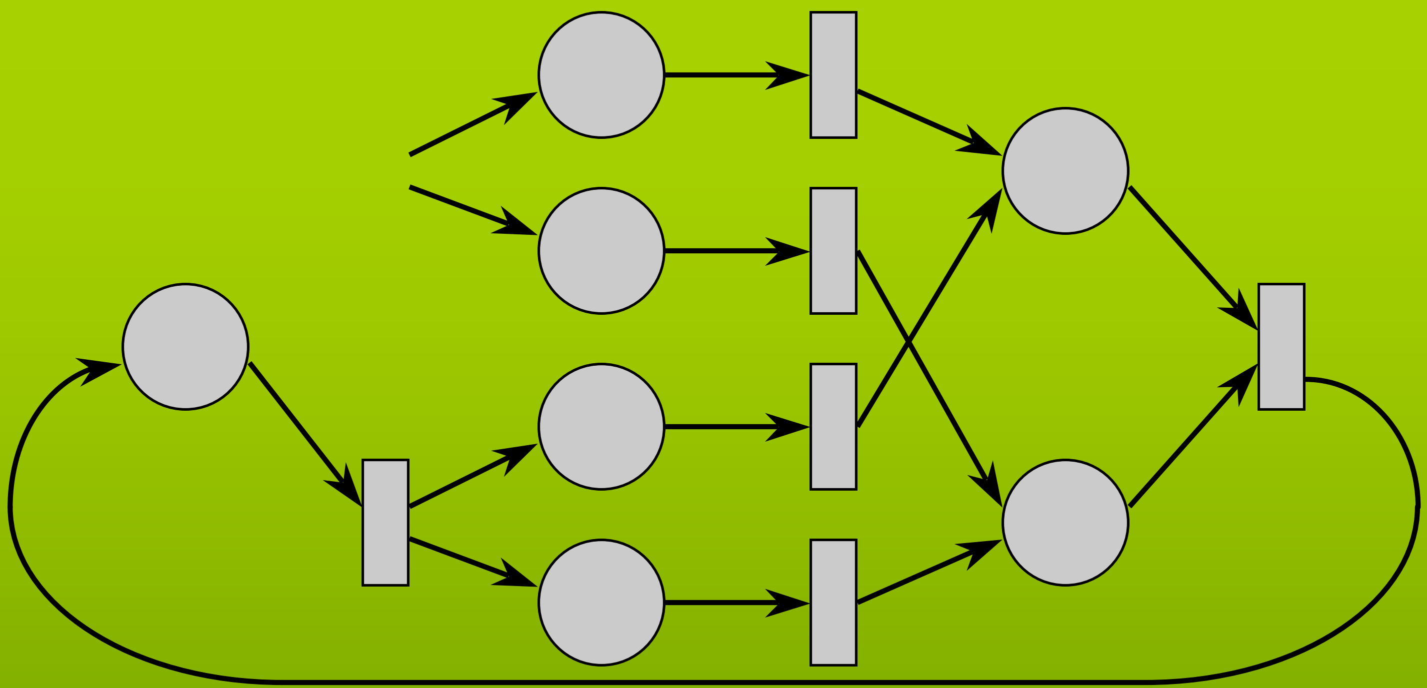

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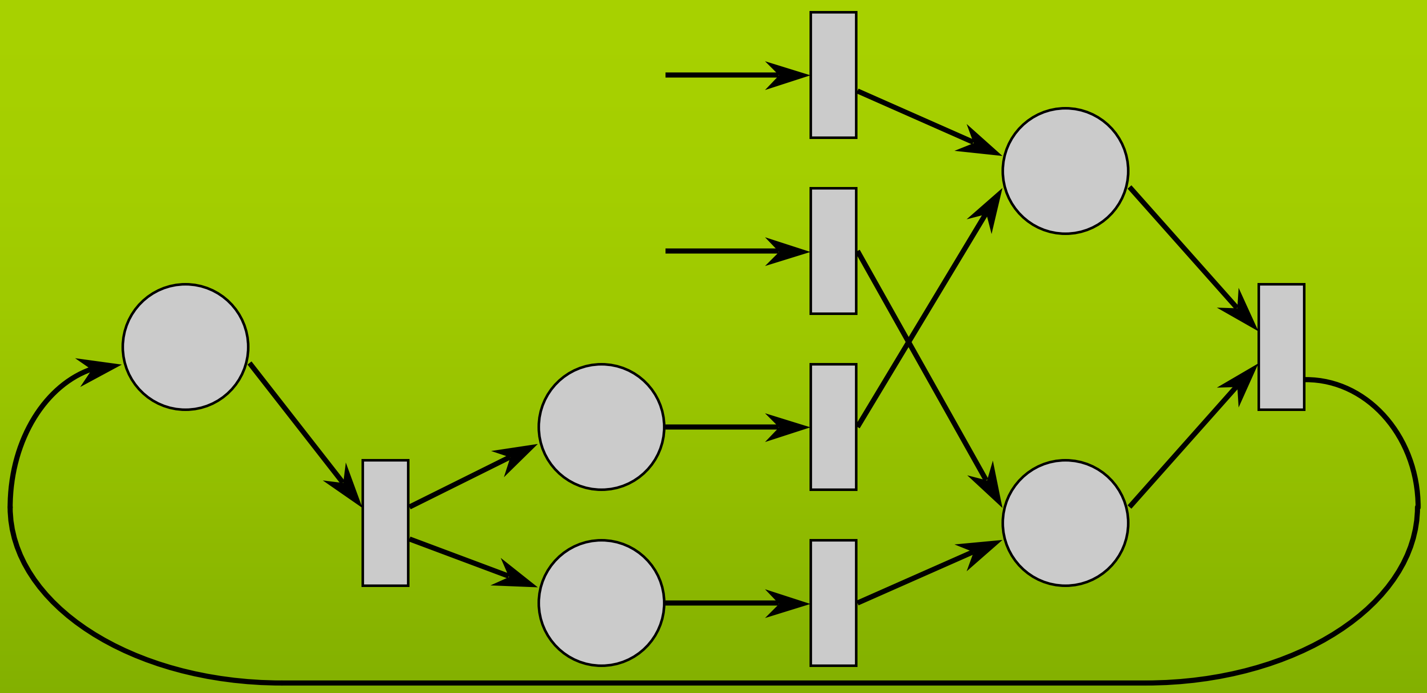

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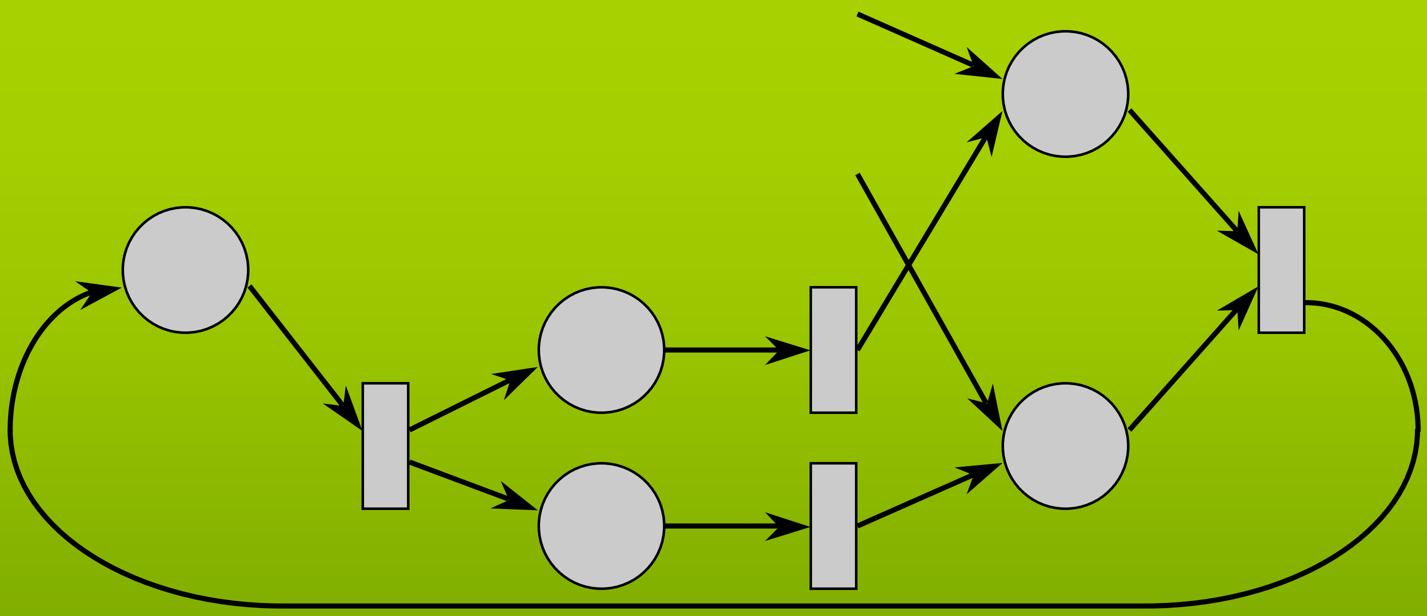

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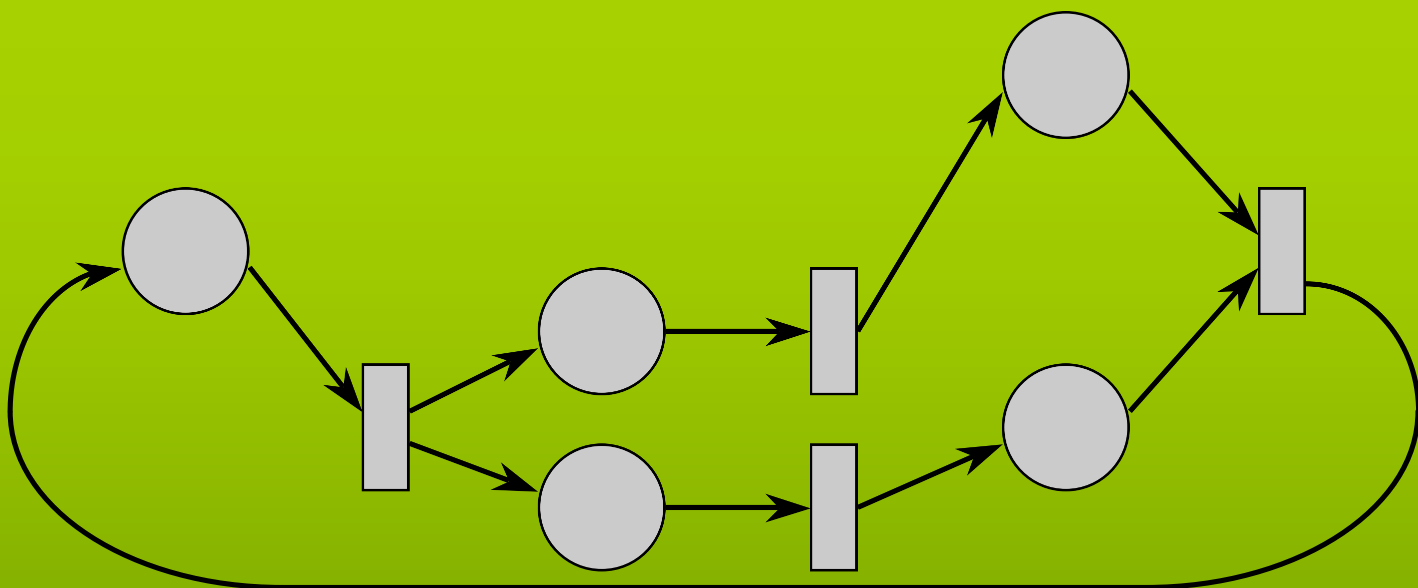
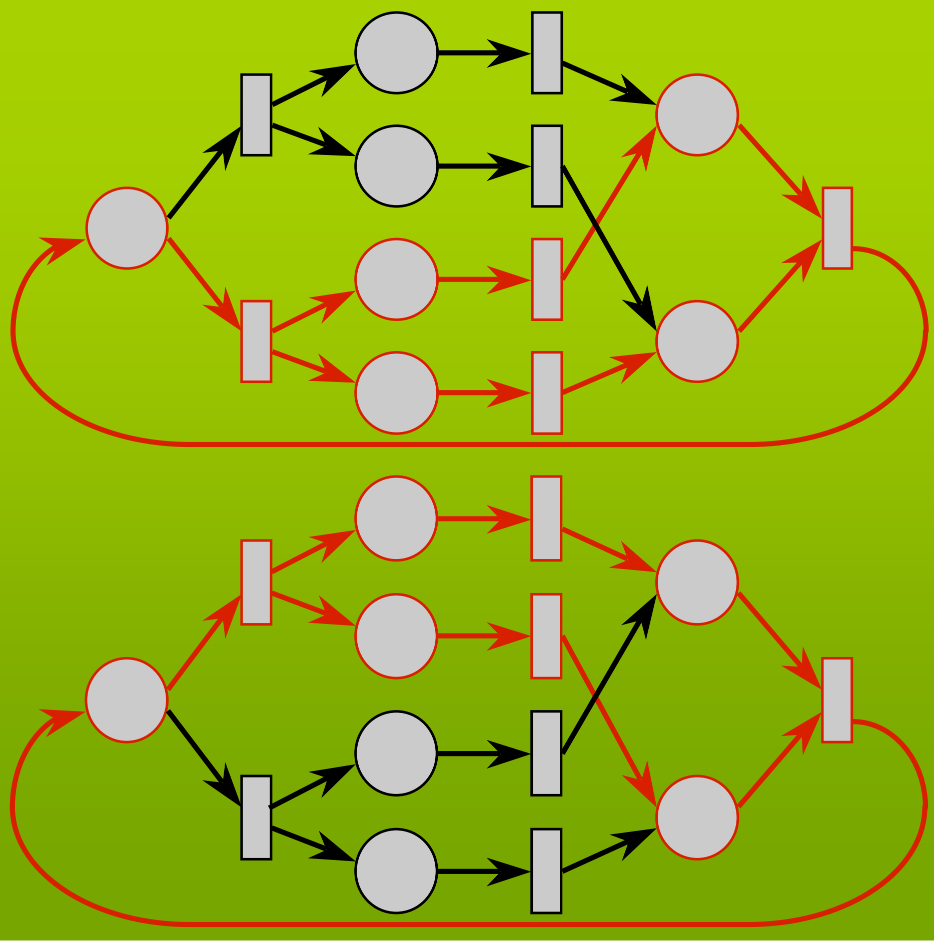


# MG reductions

- **The set of all reductions yields a cover of MG components (T-invariants)**



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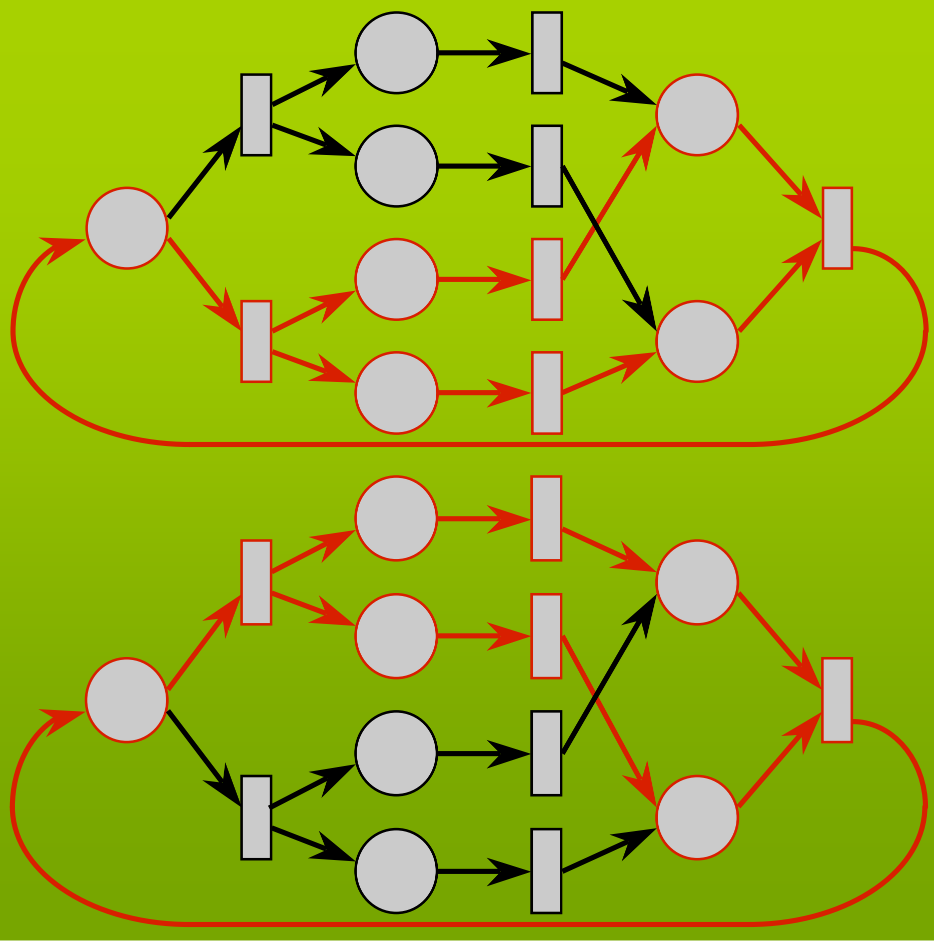

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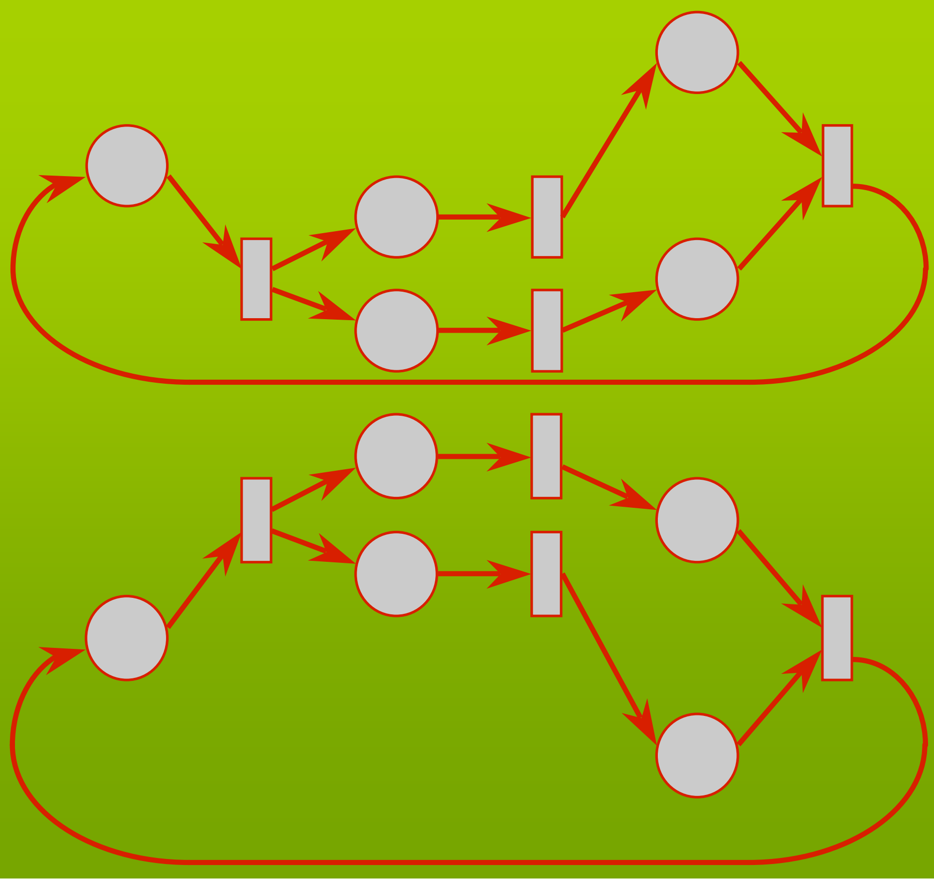


# Hack’s theorem (‘72)

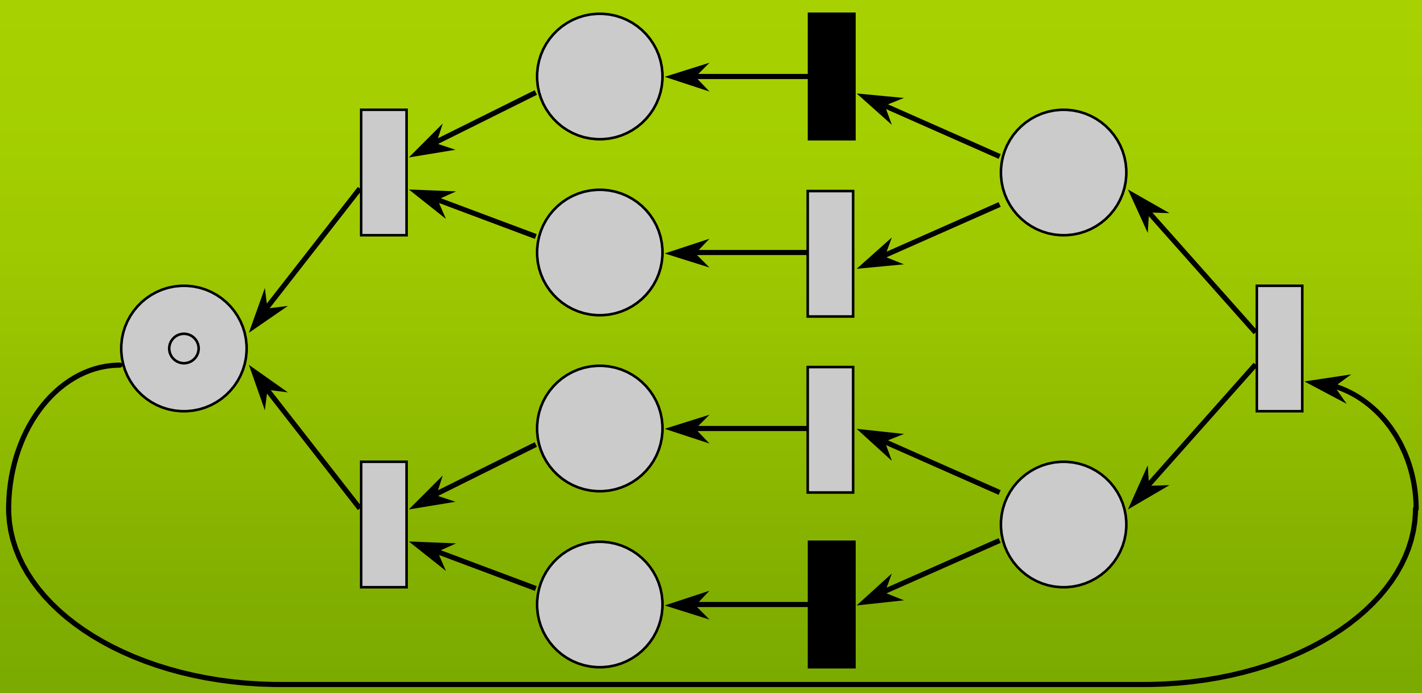
- **Let N be a Free-Choice PN:**
  - **N has a live and safe initial marking (well-formed)**

**if and only if**

  - **every MG reduction is strongly connected and not empty, and the set of all reductions covers the net**
  - **every SM reduction is strongly connected and not empty, and the set of all reductions covers the net**

# Hack's theorem

- **Example of non-live (but safe) FCN**



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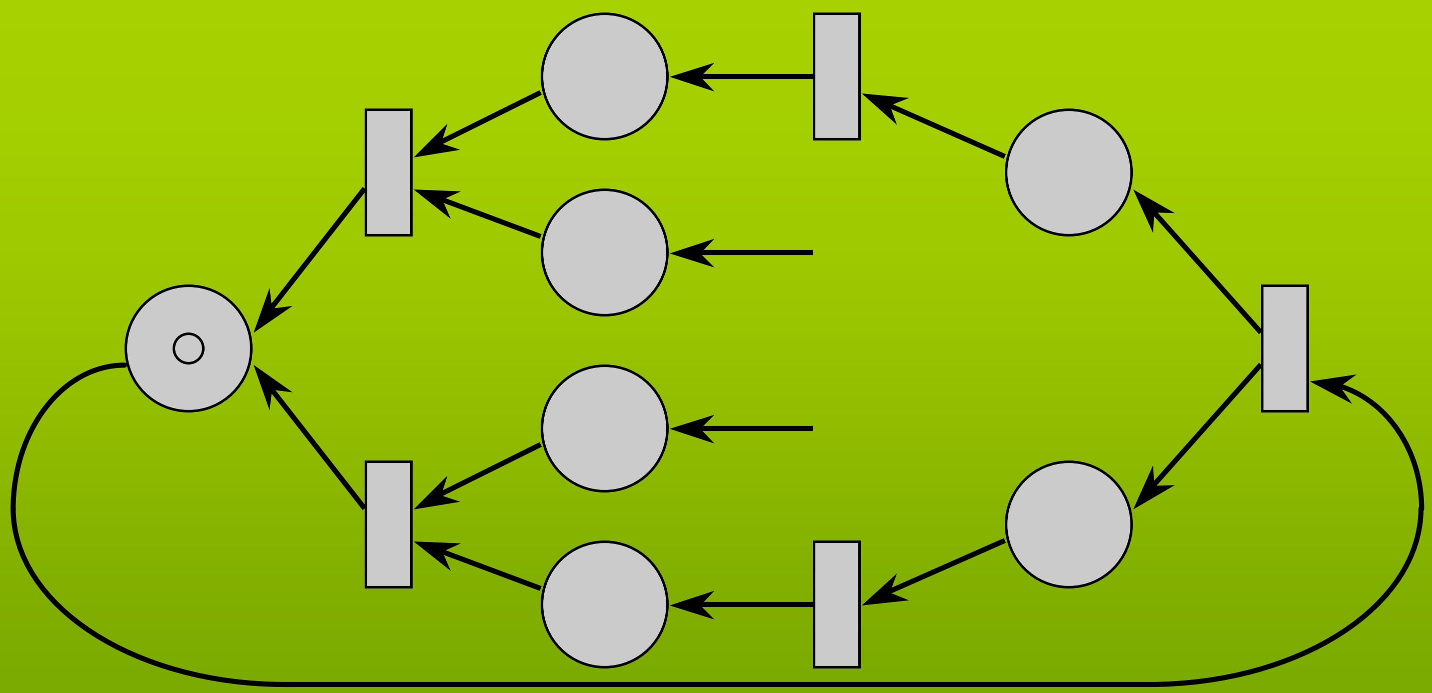

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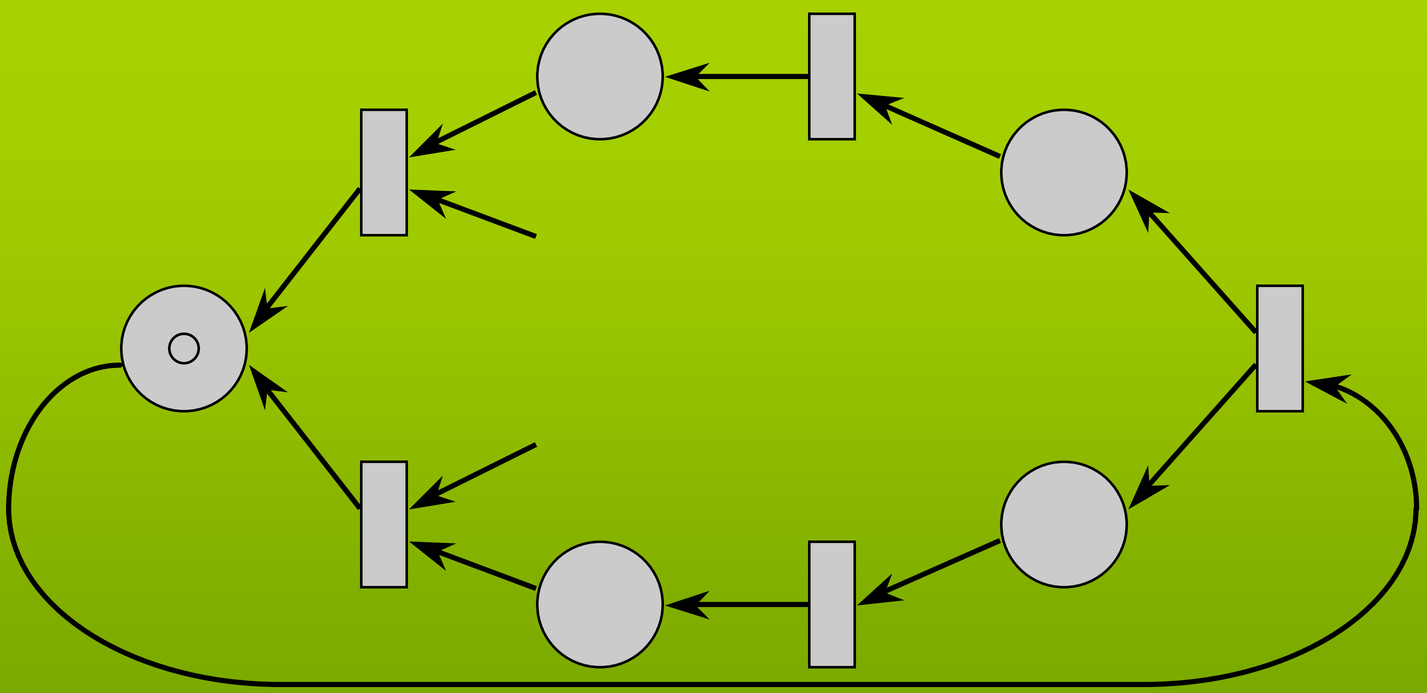

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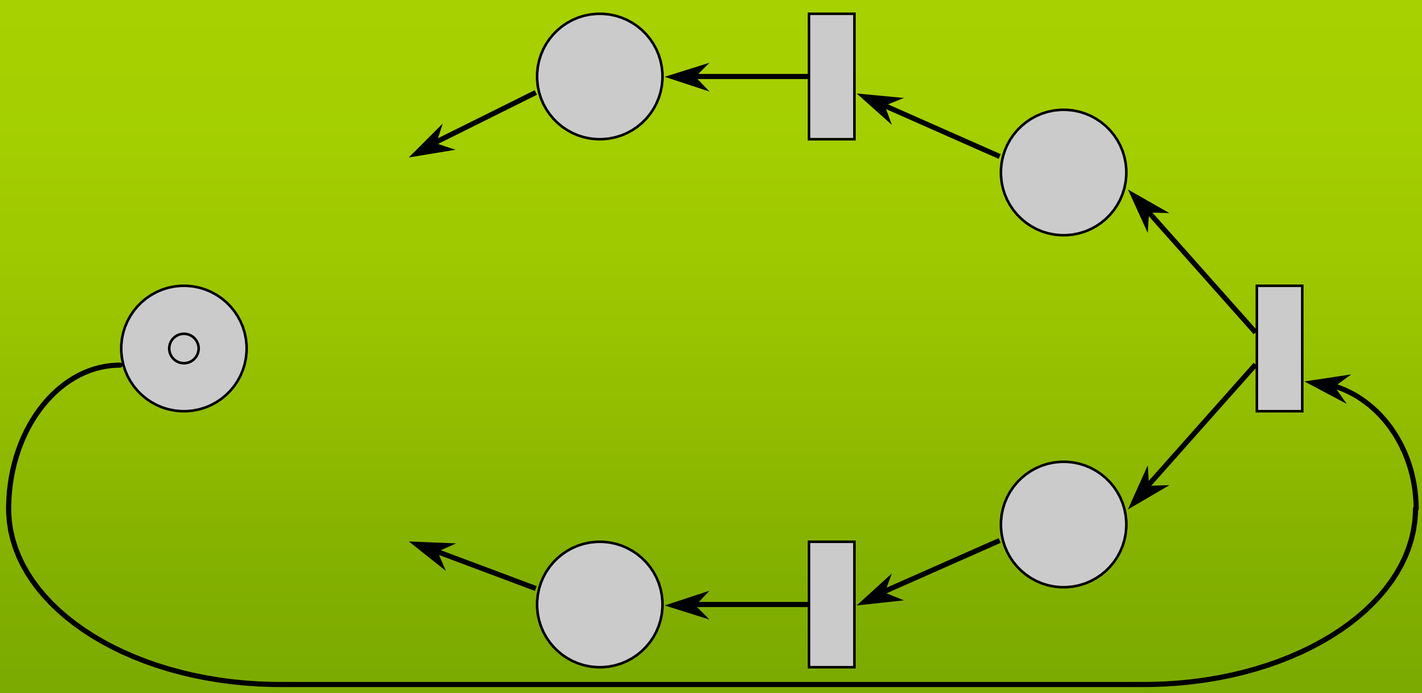

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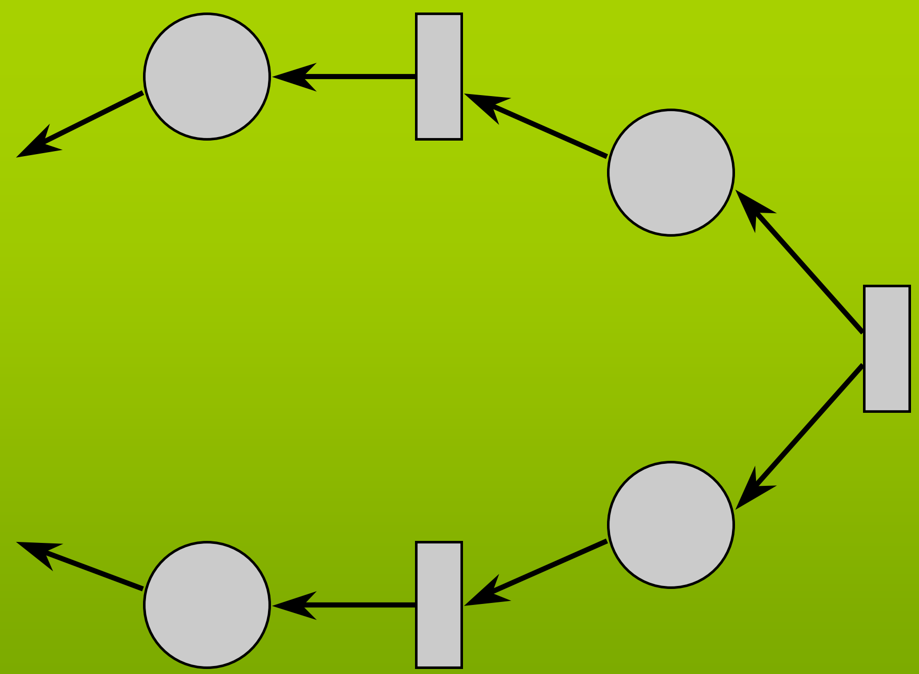

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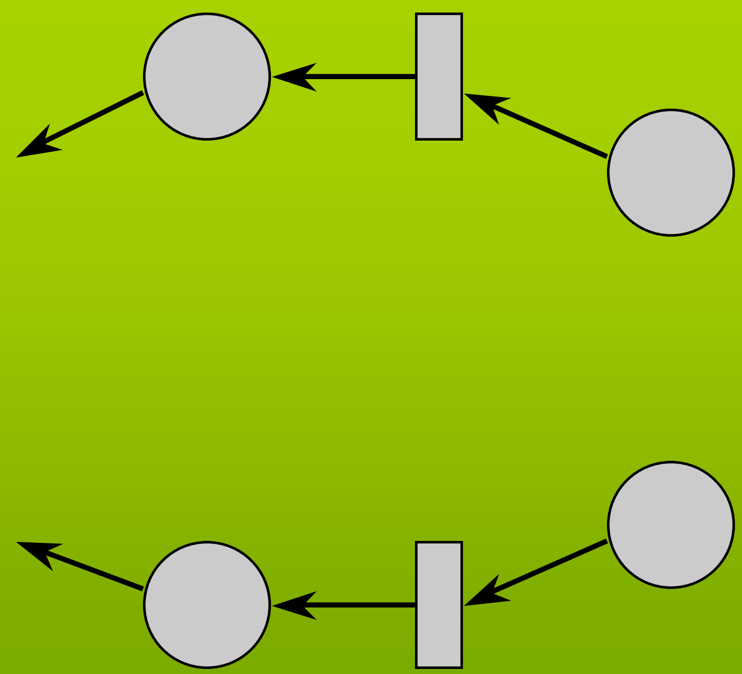

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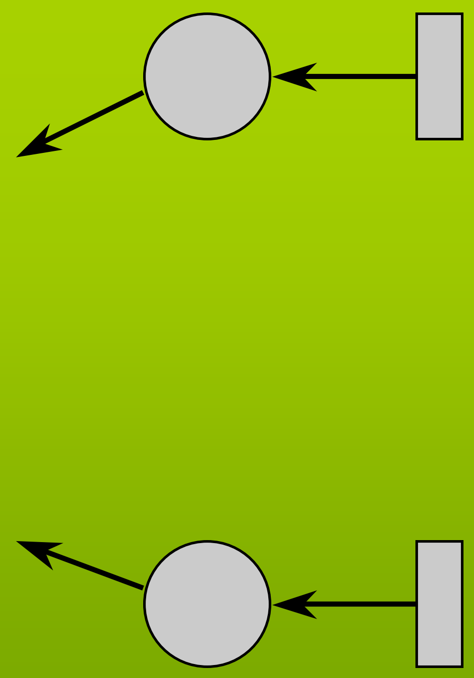

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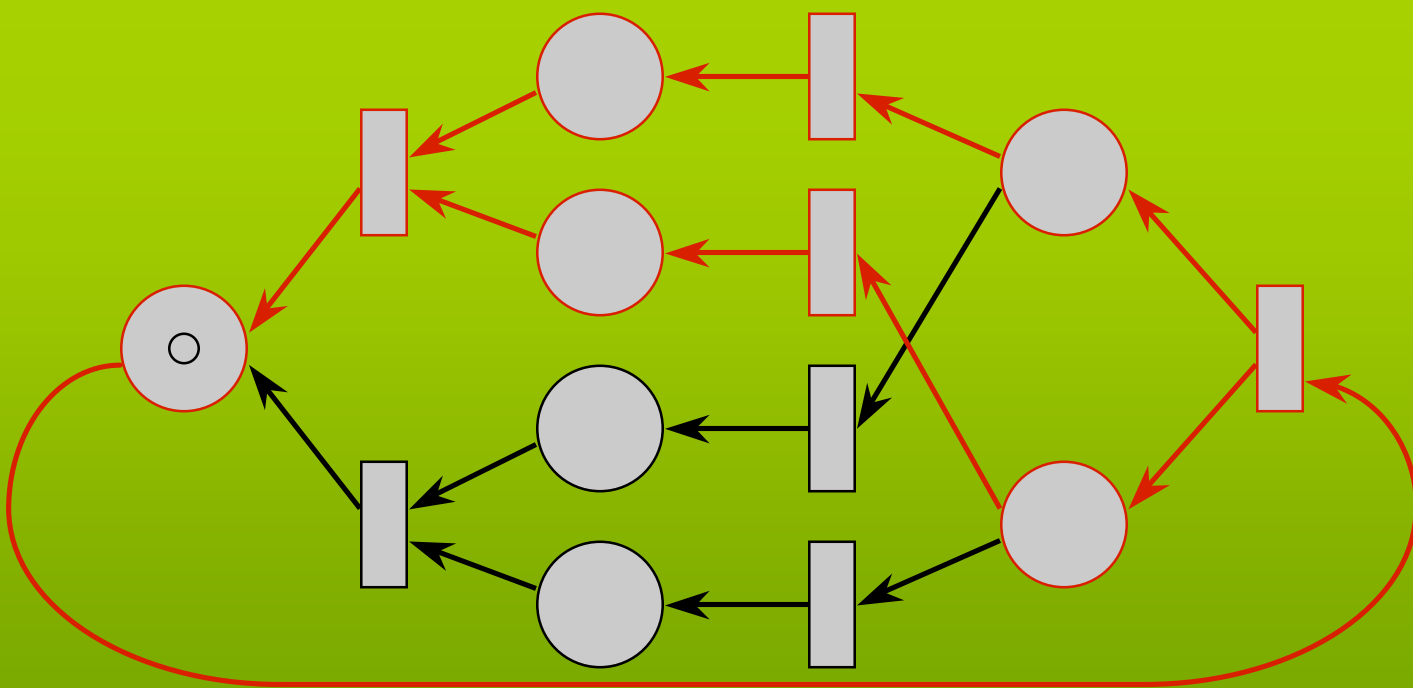

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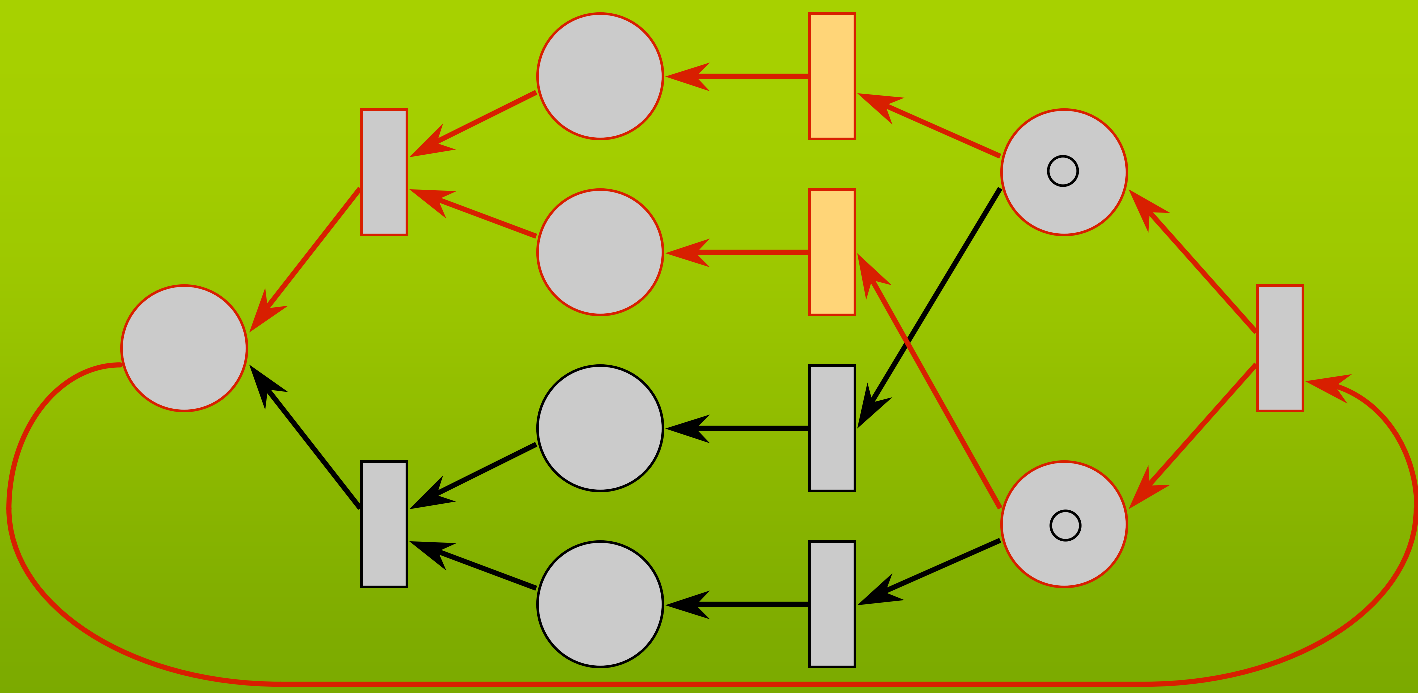

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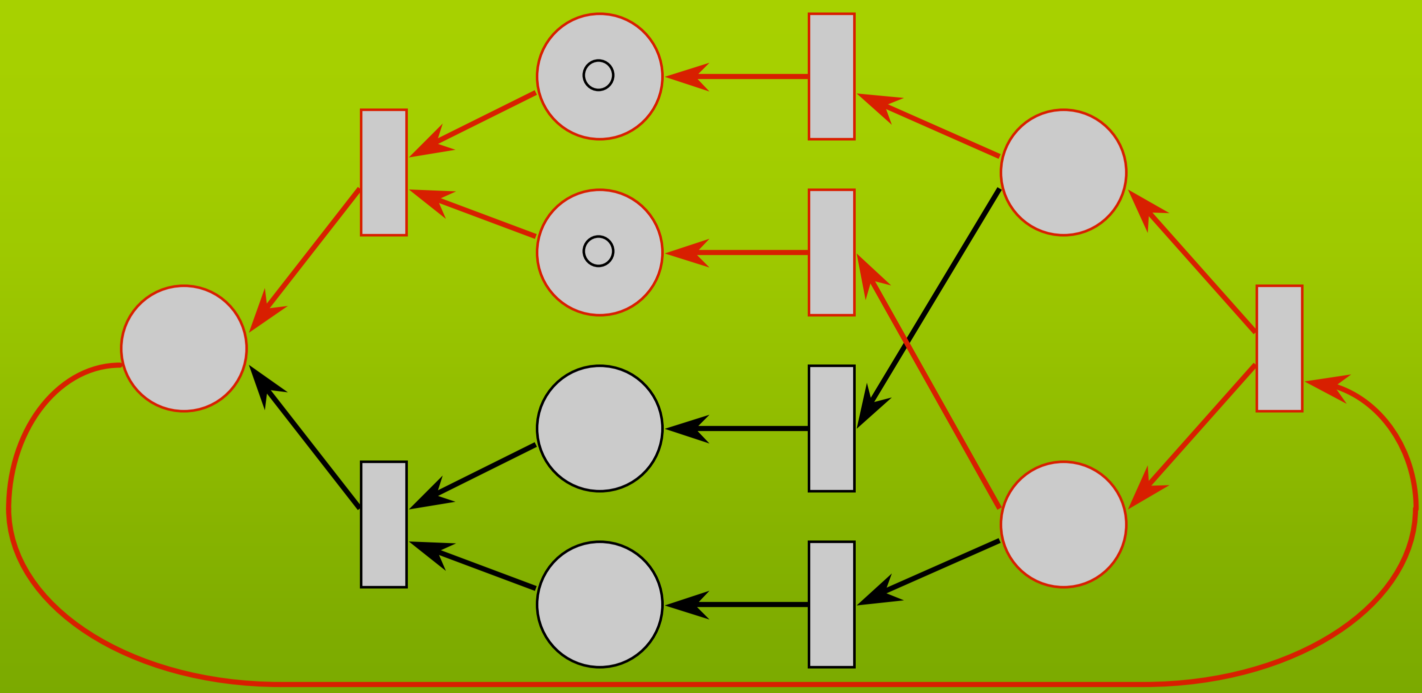

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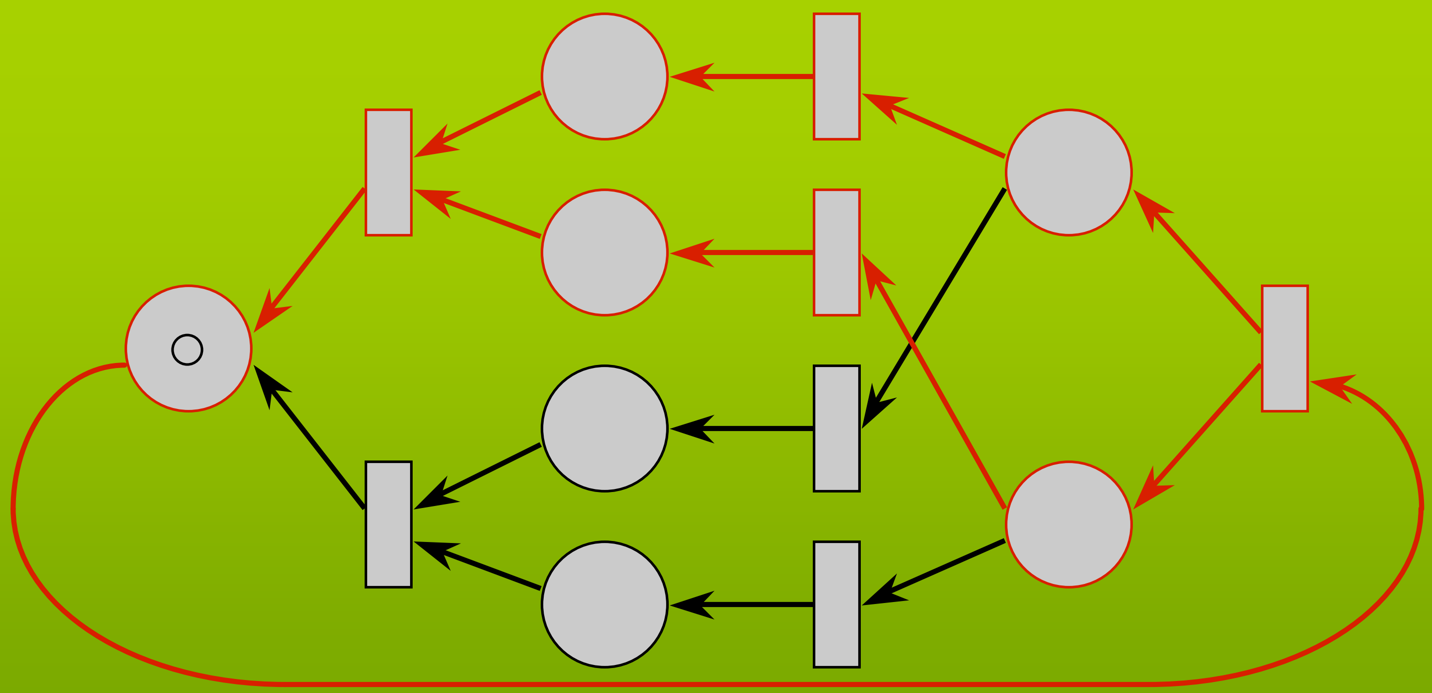

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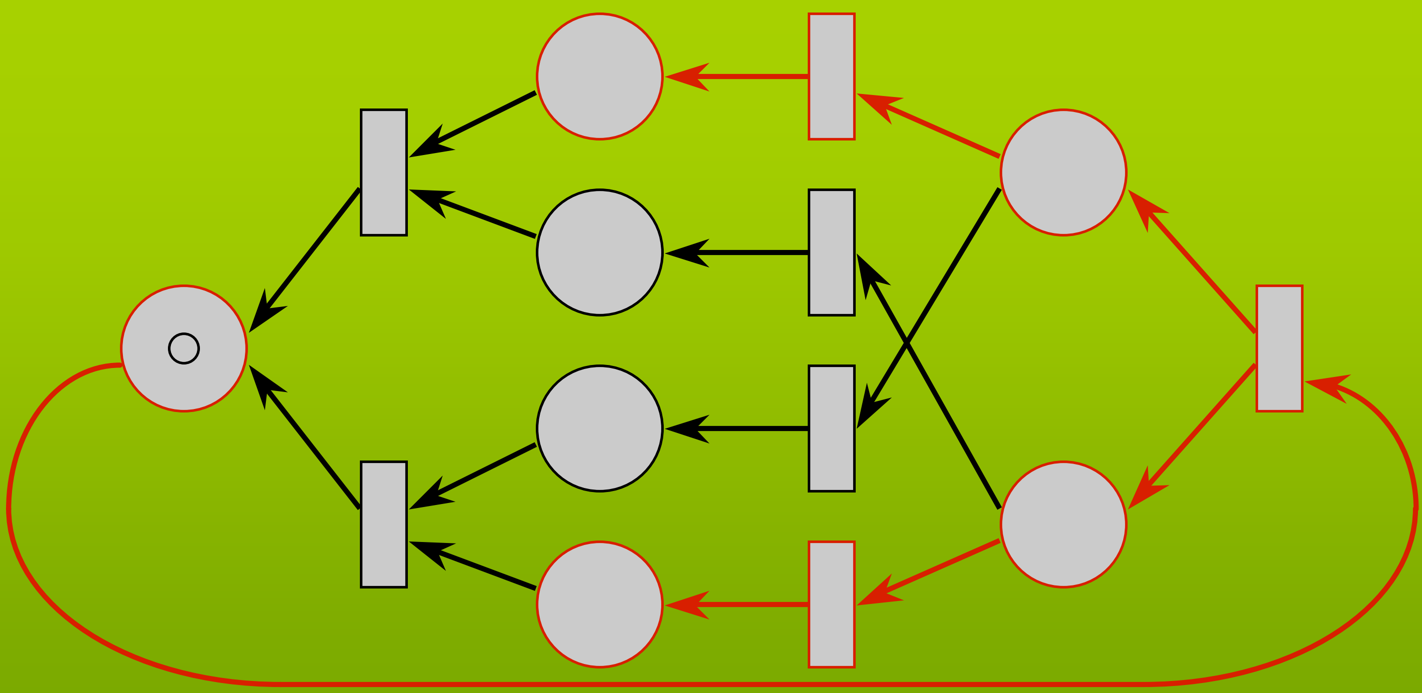

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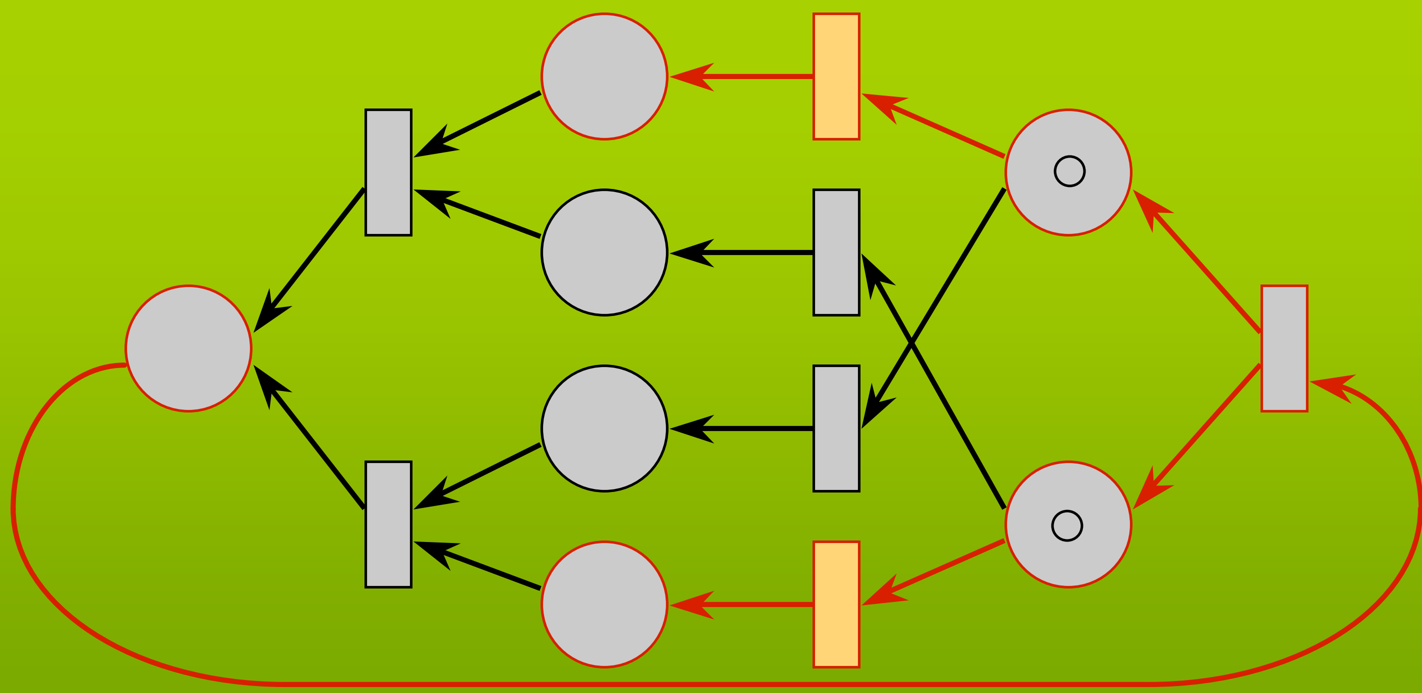

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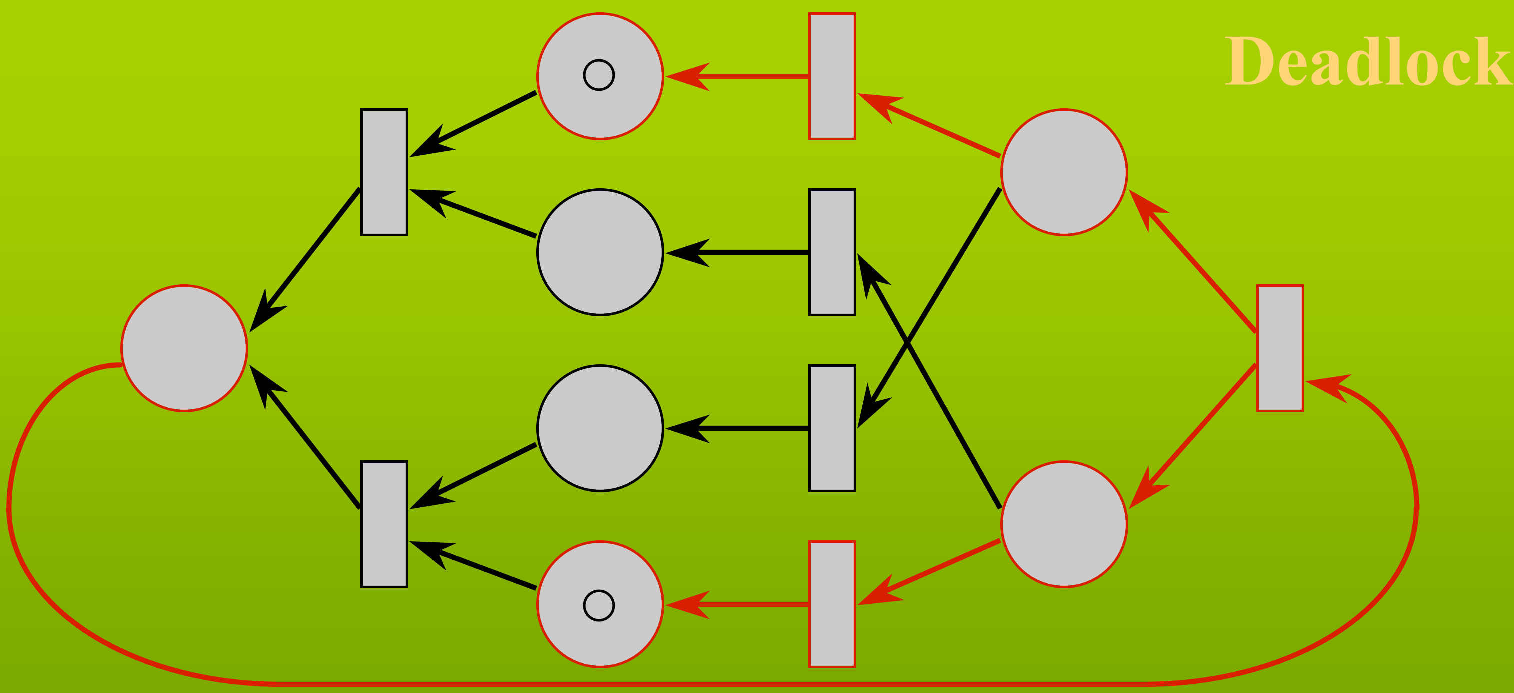


# Summary of LSFC nets

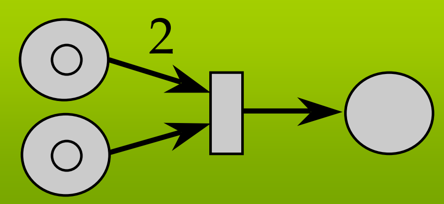
- **Largest class for which structural theory really helps**
- **Structural component analysis may be expensive**

  **(exponential number of MG and SM components in the worst case)**
- **But…**
  - **number of MG components is generally small**
  - **FC restriction simplifies characterization of behavior**

# Petri Net extensions

- Add interpretation to tokens and transitions
  - Colored nets (tokens have value)
- Add time
  - Time/timed Petri Nets (deterministic delay)
    - type (duration, delay)
    - where (place, transition)
  - Stochastic PNs (probabilistic delay)
  - Generalized Stochastic PNs (timed and immediate transitions)
- Add hierarchy
  - Place Charts Nets

# PNs Summary




- PN Graph: places (buffers), transitions (actions), tokens (data)
- Firing rule: transition enabled if there are enough tokens in each input place
- Properties
  - Structural (consistency, structural boundedness…)
  - Behavioral (reachability, boundedness, liveness…)
- Analysis techniques
  - Structural (only CN or CS): State equations, Invariants
  - Behavioral: coverability tree
- Reachability
- Subclasses: Marked Graphs, State Machines, Free-Choice PNs

# References

- T. Murata **Petri Nets: Properties, Analysis and Applications**
- http://www.daimi.au.dk/PetriNets/