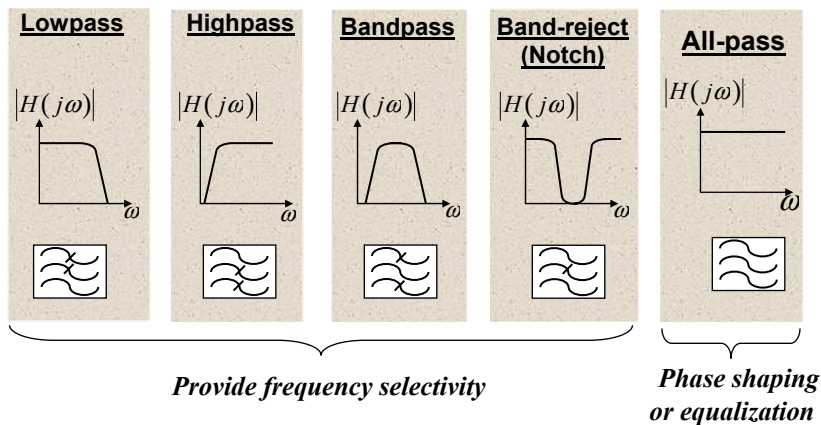


EE247 - Lecture 2 Filters

- From last lecture:
 - Dynamic range of analog circuits
- Filters:
 - Nomenclature
 - Specifications
 - Quality factor
 - Frequency characteristics
 - Group delay
 - Filter types
 - Butterworth
 - Chebyshev I & II
 - Elliptic
 - Bessel
 - Group delay comparison example
 - Biquads

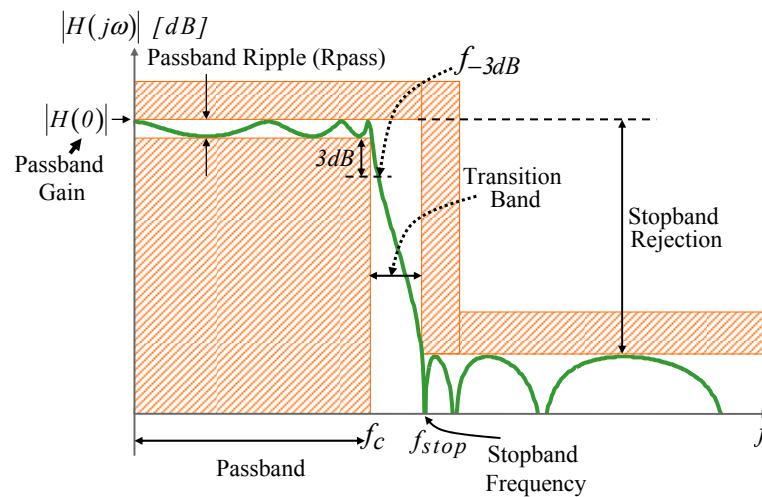
Nomenclature Filter Types wrt Frequency Range Selectivity



Filter Specifications

- Frequency characteristics (lowpass filter):
 - Passband ripple (R_{pass})
 - Cutoff frequency or $-3dB$ frequency
 - Stopband rejection
 - Passband gain
- Phase characteristics:
 - Group delay
- SNR (Dynamic range)
- SNDR (Signal to Noise+Distortion ratio)
- Linearity measures: IM3 (intermodulation distortion), HD3 (harmonic distortion), IIP3 or OIP3 (Input-referred or output-referred third order intercept point)
- Area/pole & Power/pole

Lowpass Filter Frequency Characteristics



Quality Factor (Q)

- The term quality factor (Q) has different definitions in different contexts:
 - Component quality factor (inductor & capacitor Q)
 - Pole quality factor
 - Bandpass filter quality factor
- Next 3 slides clarifies each

Component Quality Factor (Q)

- For any component with a transfer function:

$$H(j\omega) = \frac{I}{R(\omega) + jX(\omega)}$$

- Quality factor is defined as:

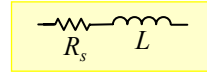
$$Q = \frac{X(\omega)}{R(\omega)} \rightarrow \frac{\text{Energy Stored}}{\text{Average Power Dissipation}} \text{ per unit time}$$

Component Quality Factor (Q) Inductor & Capacitor Quality Factor

- Inductor Q :

$$Y_L = \frac{1}{R_s + j\omega L}$$

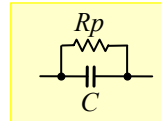
$$Q_L = \frac{\omega L}{R_s}$$



- Capacitor Q :

$$Z_C = \frac{1}{\frac{1}{R_p} + j\omega C}$$

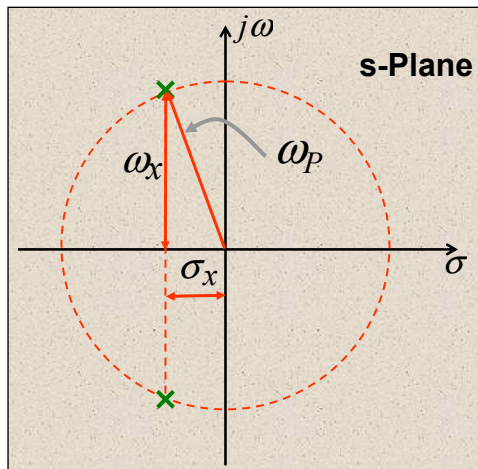
$$Q_C = \omega C R_p$$



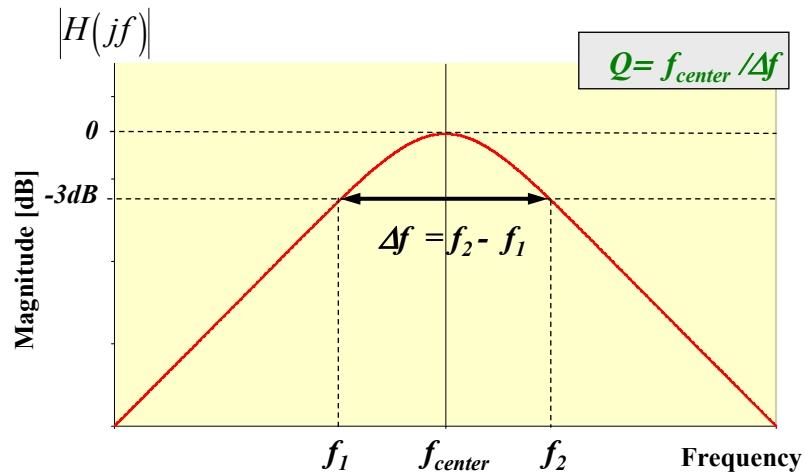
Pole Quality Factor

- Typically filter singularities include pairs of complex conjugate poles.
- Quality factor of complex conjugate poles are defined as:

$$Q_{Pole} = \frac{\omega_p}{2\sigma_x}$$



Bandpass Filter Quality Factor (Q)



What is Group Delay?

- Consider a continuous time filter with s-domain transfer function $G(s)$:

$$G(j\omega) \equiv |G(j\omega)| e^{j\theta(\omega)}$$

- Let us apply a signal to the filter input composed of sum of two sinewaves at slightly different frequencies ($\Delta\omega \ll \omega$):

$$v_{IN}(t) = A_1 \sin(\omega t) + A_2 \sin[(\omega + \Delta\omega) t]$$

- The filter output is:

$$v_{OUT}(t) = A_1 |G(j\omega)| \sin[\omega t + \theta(\omega)] + A_2 |G[j(\omega + \Delta\omega)]| \sin[(\omega + \Delta\omega)t + \theta(\omega + \Delta\omega)]$$

What is Group Delay?

$$v_{\text{OUT}}(t) = A_1 |G(j\omega)| \sin \left\{ \omega \left[t + \frac{\theta(\omega)}{\omega} \right] \right\} +$$

$$+ A_2 |G[j(\omega+\Delta\omega)]| \sin \left\{ (\omega+\Delta\omega) \left[t + \frac{\theta(\omega+\Delta\omega)}{\omega+\Delta\omega} \right] \right\}$$

Since $\frac{\Delta\omega}{\omega} \ll 1$ then $\left[\frac{\Delta\omega}{\omega}\right]^2 \rightarrow 0$

$$\frac{\theta(\omega+\Delta\omega)}{\omega+\Delta\omega} \cong \left[\theta(\omega) + \frac{d\theta(\omega)}{d\omega} \Delta\omega \right] \left[\frac{1}{\omega} \left(1 - \frac{\Delta\omega}{\omega} \right) \right]$$

$$\cong \frac{\theta(\omega)}{\omega} + \left(\frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega} \right) \frac{\Delta\omega}{\omega}$$

What is Group Delay? Signal Magnitude and Phase Impairment

$$v_{\text{OUT}}(t) = A_1 |G(j\omega)| \sin \left\{ \omega \left[t + \frac{\theta(\omega)}{\omega} \right] \right\} +$$

$$+ A_2 |G[j(\omega+\Delta\omega)]| \sin \left\{ (\omega+\Delta\omega) \left[t + \frac{\theta(\omega)}{\omega} + \underbrace{\left(\frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega} \right) \frac{\Delta\omega}{\omega}} \right] \right\}$$

- $\tau_{\text{PD}} \equiv -\theta(\omega)/\omega$ is called the "phase delay" and has units of time
- If the second term is zero, then the filter's output at frequency $\omega+\Delta\omega$ and the output at frequency ω are each delayed in time by $-\theta(\omega)/\omega$
- If the second term in the phase of the 2nd sin wave is non-zero, then the filter's output at frequency $\omega+\Delta\omega$ is time-shifted differently than the filter's output at frequency ω
→ "Phase distortion"

What is Group Delay? Signal Magnitude and Phase Impairment

- Phase distortion is avoided only if:

$$\frac{d\theta(\omega)}{d\omega} - \frac{\theta(\omega)}{\omega} = 0$$

- Clearly, if $\theta(\omega)=k\omega$, k a constant, \rightarrow no phase distortion
- This type of filter phase response is called “linear phase”
 \rightarrow Phase shift varies linearly with frequency
- $\tau_{GR} \equiv -d\theta(\omega)/d\omega$ is called the “group delay” and also has units of time. For a linear phase filter $\tau_{GR} \equiv \tau_{PD} = k$
 $\rightarrow \tau_{GR} = \tau_{PD}$ implies linear phase
- Note: Filters with $\theta(\omega)=k\omega+C$ are also called linear phase filters, but they’re not free of phase distortion

What is Group Delay? Signal Magnitude and Phase Impairment

- If $\tau_{GR} = \tau_{PD} \rightarrow$ No phase distortion

$$\begin{aligned} v_{OUT}(t) = & A_1 |G(j\omega)| \sin \left[\omega (t - \tau_{GR}) \right] + \\ & + A_2 |G[j(\omega+\Delta\omega)]| \sin \left[(\omega+\Delta\omega) (t - \tau_{GR}) \right] \end{aligned}$$

- If also $|G(j\omega)| = |G[j(\omega+\Delta\omega)]|$ for all input frequencies within the signal-band, v_{OUT} is a scaled, time-shifted replica of the input, with no “signal magnitude distortion” :
- In most cases neither of these conditions are exactly realizable

Summary Group Delay

- Phase delay is defined as:

$$\tau_{PD} \equiv -\theta(\omega)/\omega \quad [\text{time}]$$

- Group delay is defined as :

$$\tau_{GR} \equiv -d\theta(\omega)/d\omega \quad [\text{time}]$$

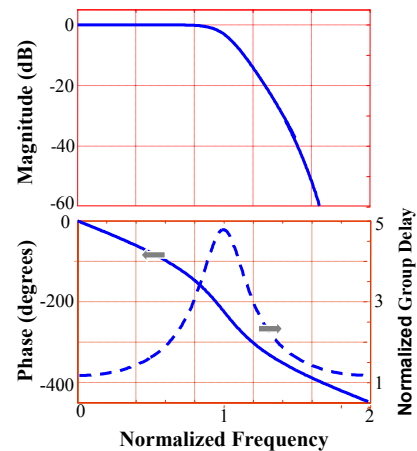
- If $\theta(\omega)=k\omega$, k a constant, \rightarrow no phase distortion
- For a linear phase filter $\tau_{GR} \equiv \tau_{PD} = k$

Filter Types wrt Frequency Response Lowpass Butterworth Filter

- Maximally flat amplitude within the filter passband

$$\left. \frac{d^N |H(j\omega)|}{d\omega} \right|_{\omega=0} = 0$$

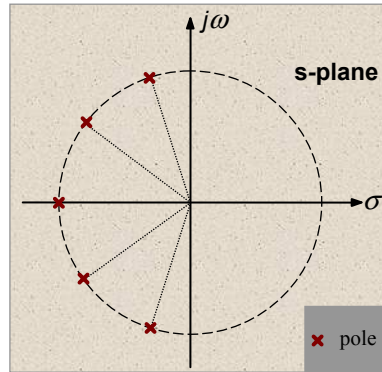
- Moderate phase distortion



Example: 5th Order Butterworth filter

Lowpass Butterworth Filter

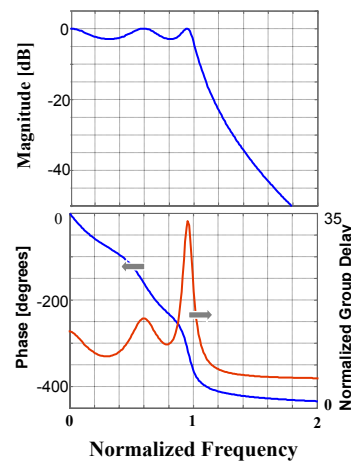
- All poles
- Number of poles equal to filter order
- Poles located on the unit circle with equal angles



Example: 5th Order Butterworth Filter

Filter Types Chebyshev I Lowpass Filter

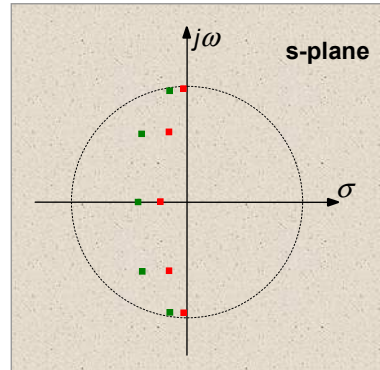
- Chebyshev I filter
 - Ripple in the passband
 - Sharper transition band compared to Butterworth (for the same number of poles)
 - Poorer group delay compared to Butterworth
 - More ripple in passband \rightarrow poorer phase response



Example: 5th Order Chebyshev filter

Chebyshev I Lowpass Filter Characteristics

- All poles
- Poles located on an ellipse inside the unit circle
- Allowing more ripple in the passband:
 - ⇒ Narrower transition band
 - ⇒ Sharper cut-off
 - ⇒ Higher pole Q
 - ⇒ Poorer phase response

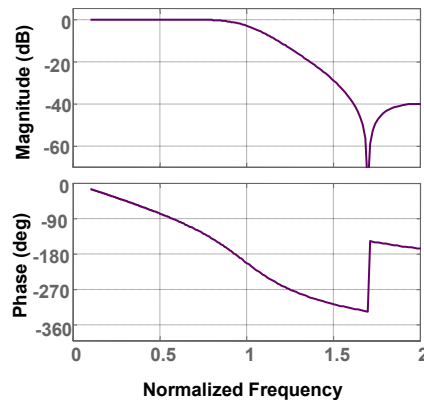


- Chebyshev I LPF 3dB passband ripple
- Chebyshev I LPF 0.1dB passband ripple

Example: 5th Order Chebyshev I Filter

Filter Types Chebyshev II Lowpass

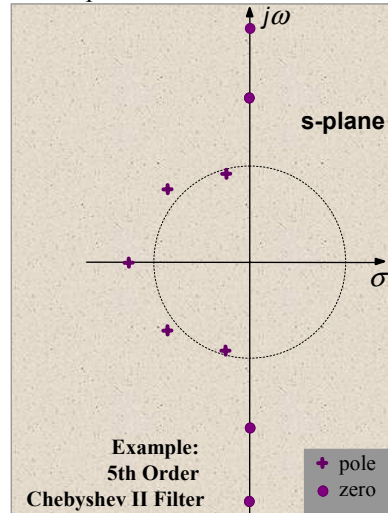
- Chebyshev II filter
 - No ripple in passband
 - Nulls or notches in stopband
 - Sharper transition band compared to Butterworth
 - Passband phase more linear compared to Chebyshev I



Example: 5th Order Chebyshev II filter

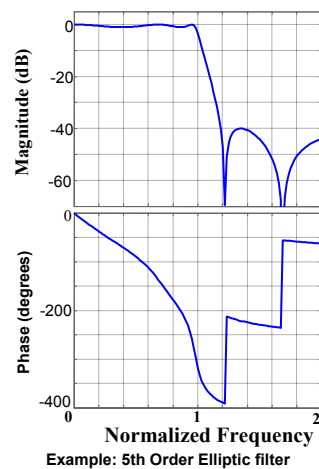
Filter Types Chebyshev II Lowpass

- Poles & finite zeros
 - No. of poles n
($n \rightarrow$ filter order)
 - No. of finite zeros: $n-1$
- Poles located both inside & outside of the unit circle
- Complex conjugate zeros located on $j\omega$ axis
- Zeros create nulls in stopband



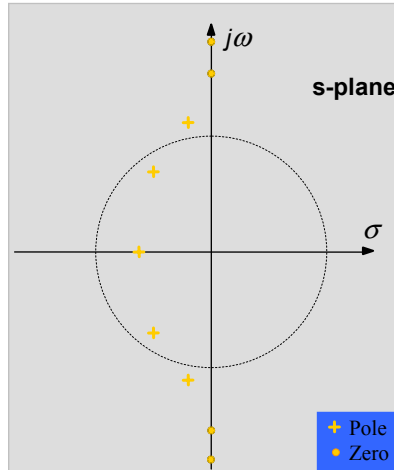
Filter Types Elliptic Lowpass Filter

- Elliptic filter
 - Ripple in passband
 - Nulls in the stopband
 - Sharper transition band compared to Butterworth & both Chebyshevs
 - Poorest phase response



Filter Types Elliptic Lowpass Filter

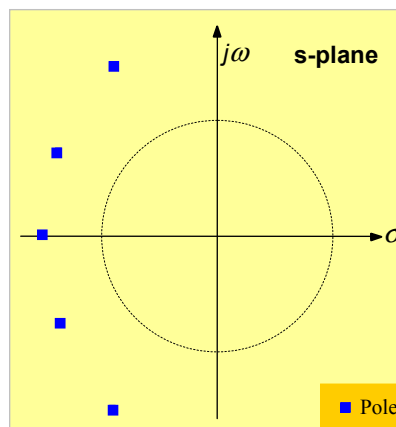
- Poles & finite zeros
 - No. of poles: n
 - No. of finite zeros: $n-1$
- Zeros located on $j\omega$ axis
- Sharp cut-off
 - ⇒ Narrower transition band
 - ⇒ Pole Q higher compared to the previous filter types



Example: 5th Order Elliptic Filter

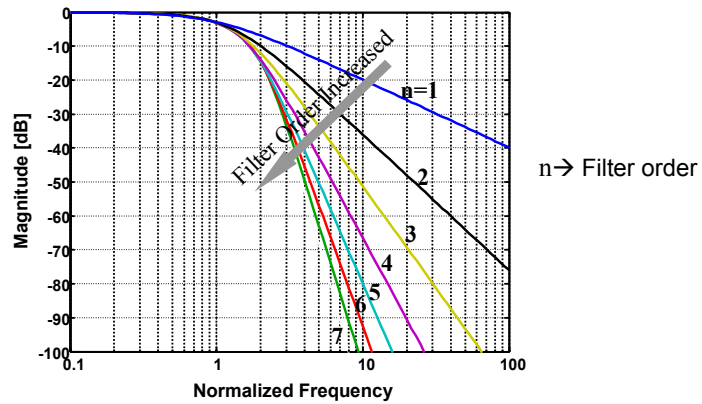
Filter Types Bessel Lowpass Filter

- Bessel
 - All poles
 - Poles outside unit circle
 - Relatively low Q poles
 - Maximally flat group delay
 - Poor out-of-band attenuation

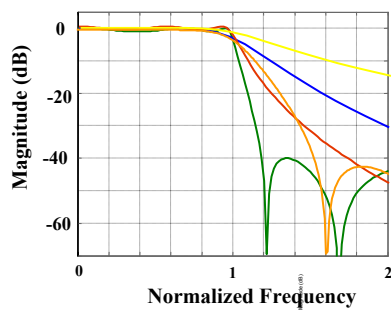


Example: 5th Order Bessel filter

Magnitude Response Behaviour as a Function of Filter Order Example: Bessel Filter



Filter Types Comparison of Various Type LPF Magnitude Response

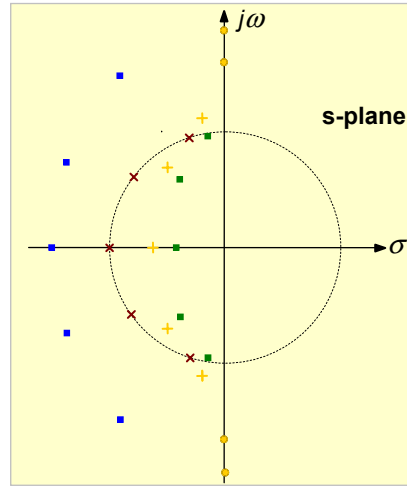


All 5th order filters with same corner freq.

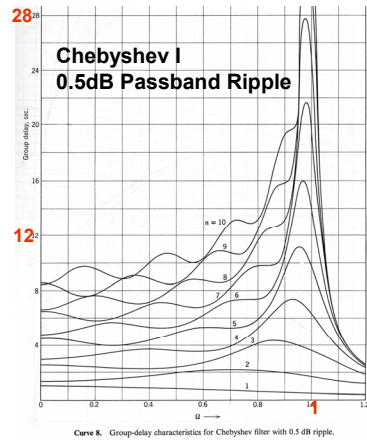
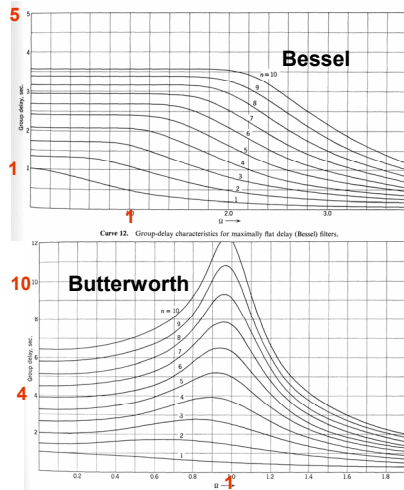
Bessel ———
 Butterworth ———
 Chebyshev I ———
 Chebyshev II ———
 Elliptic ———

Filter Types Comparison of Various LPF Singularities

- Poles Bessel
- × Poles Butterworth
- + Poles Elliptic
- Zeros Elliptic
- Poles Chebyshev I 0.1dB



Comparison of Various LPF Groupdelay

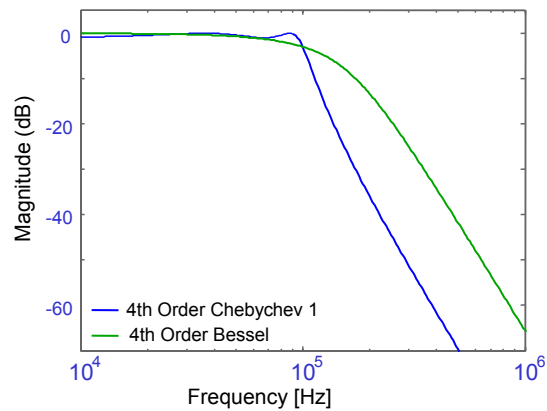


Ref: A. Zverev, *Handbook of filter synthesis*, Wiley, 1967.

Group Delay Comparison Example

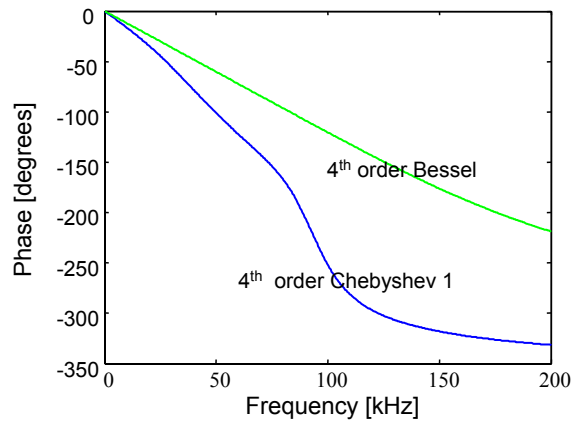
- Lowpass filter with 100kHz corner frequency
- Chebyshev I versus Bessel
 - Both filters 4th order- same $-3dB$ point
 - Passband ripple of $1dB$ allowed for Chebyshev I

Magnitude Response 4th Order Chebyshev I versus Bessel



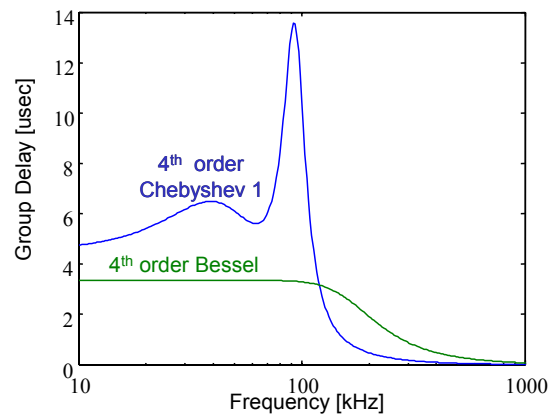
Phase Response

4th Order Chebyshev I versus Bessel

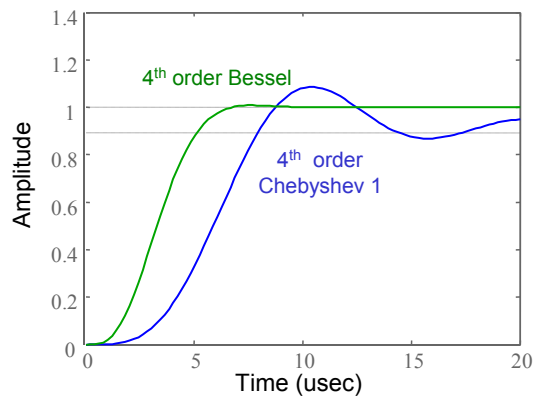


Group Delay

4th Order Chebyshev I versus Bessel



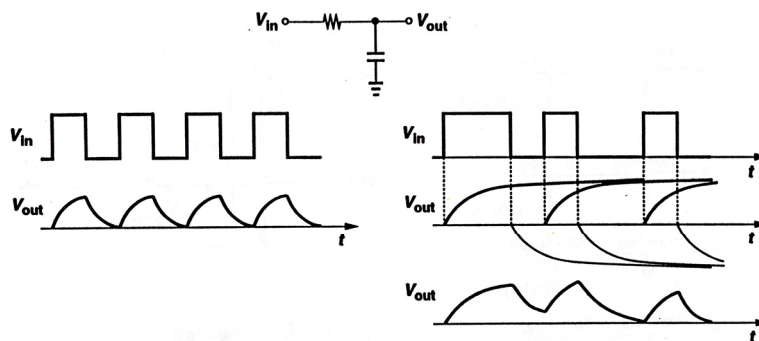
Step Response 4th Order Chebyshev I versus Bessel



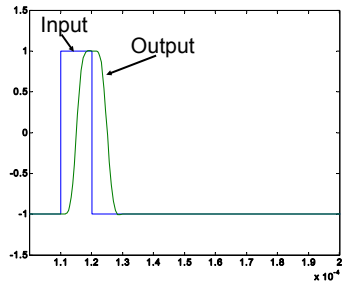
Intersymbol Interference (ISI)

ISI → Broadening of pulses resulting in interference between successive transmitted pulses

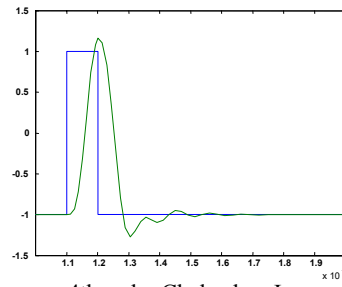
Example: Simple RC filter



Pulse Impairment Bessel versus Chebyshev



4th order Bessel



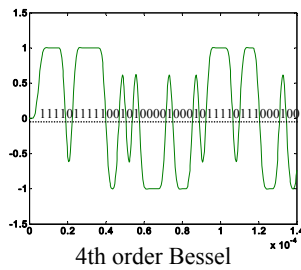
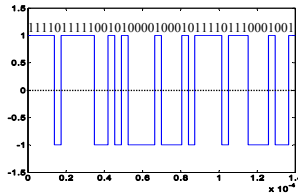
4th order Chebyshev I

Note that in the case of the Chebyshev filter not only the pulse has broadened but it also has a long tail

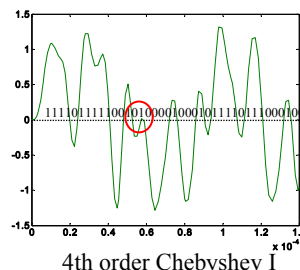
→ More ISI for Chebyshev compared to Bessel

Response to Pseudo-Random Data Chebyshev versus Bessel

Input Signal:
Symbol rate 1/130kHz →



4th order Bessel



4th order Chebyshev I

Summary Filter Types

- Filters with high signal attenuation per pole \Rightarrow poor phase response
- For a given signal attenuation, requirement of preserving constant groupdelay \rightarrow Higher order filter
 - In the case of passive filters \Rightarrow higher component count
 - For integrated active filters \Rightarrow higher chip area & power dissipation
- In cases where filter is followed by ADC and DSP
 - Possible to digitally correct for phase impairments incurred by the analog circuitry by using digital phase equalizers & thus reducing the required filter order

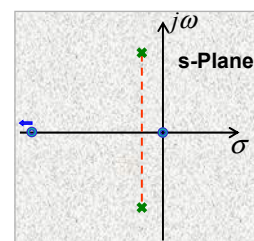
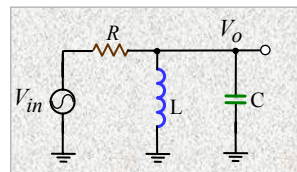
RLC Filters

- Bandpass filter (2nd order):

$$\frac{V_o}{V_{in}} = \frac{sRC}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$$\omega_0 = 1/\sqrt{LC}$$

$$Q = \omega_0 RC = \frac{R}{L\omega_0}$$

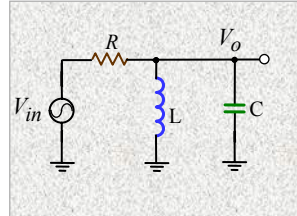


Singularities: Pair of complex conjugate poles
Zeros @ $f=0$ & $f=inf$.

RLC Filters Example

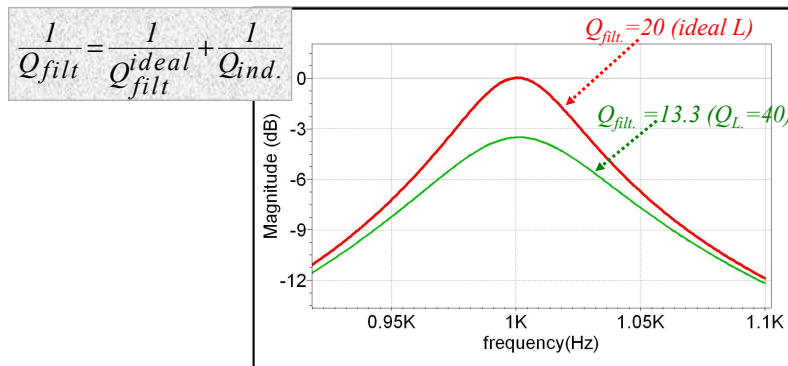
- Design a bandpass filter with:

- Center frequency of 1kHz
- Quality factor of 20



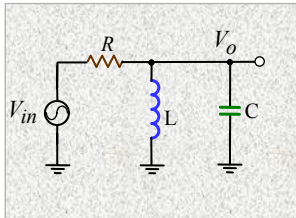
- First assume the inductor is ideal
- Next consider the case where the inductor has series R resulting in a finite inductor Q of 40
- What is the effect of finite inductor Q on the overall filter Q?

RLC Filters Effect of Finite Component Q



⇒ **Need to have component Q much higher compared to desired filter Q**

RLC Filters



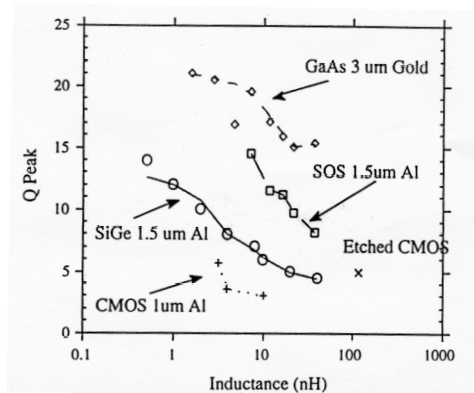
Question:
Can RLC filters be integrated on-chip?

Monolithic Inductors Feasible Quality Factor & Value

Typically, on-chip inductors built as spiral structures out of metal/s layers

$$Q_L = (\omega L/R)$$

Q_L measured at frequencies of operation ($>1\text{GHz}$)



⇒ Feasible monolithic inductor in CMOS tech. $<10\text{nH}$ with $Q < 7$

❖Ref: "Radio Frequency Filters", Lawrence Larson; Mead workshop presentation 1999

Integrated Filters

- Implementation of RLC filters in CMOS technologies requires on-chip inductors
 - Integrated $L < 10\text{nH}$ with $Q < 10$
 - Combined with max. cap. 20pF
 - *LC filters in the monolithic form feasible: freq > 350MHz*
 - *(Learn more in EE242 & RF circuit courses)*
- Analog/Digital interface circuitry require fully integrated filters with critical frequencies $\ll 350\text{MHz}$
- Hence:

⇒ **Need to build active filters built without inductor**

Filters

2nd Order Transfer Functions (Biquads)

- Biquadratic (2nd order) transfer function:

$$H(s) = \frac{1}{1 + \frac{s}{\omega_p Q_p} + \frac{s^2}{\omega_p^2}}$$

$$|H(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_p^2}\right)^2 + \left(\frac{\omega}{\omega_p Q_p}\right)^2}} \longrightarrow \begin{cases} |H(j\omega)|_{\omega=0} = 1 \\ |H(j\omega)|_{\omega \rightarrow \infty} = 0 \\ |H(j\omega)|_{\omega=\omega_p} = Q_p \end{cases}$$

$$\text{Biquad poles @: } s = -\frac{\omega_p}{2Q_p} \left(1 \pm \sqrt{1 - 4Q_p^2}\right)$$

Note: for $Q_p \leq \frac{1}{2}$ poles are real, complex otherwise

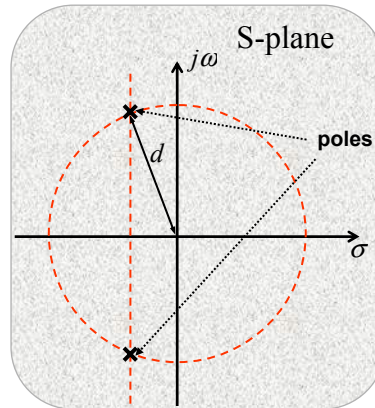
Biquad Complex Poles

$Q_P > \frac{1}{2} \rightarrow$ Complex conjugate poles:

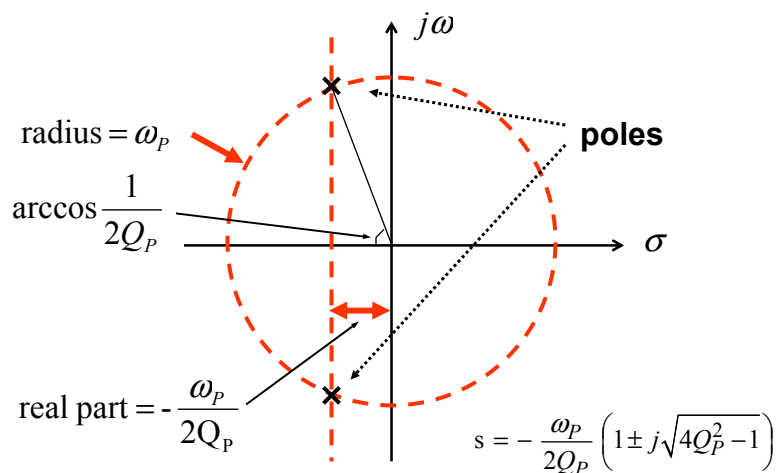
$$s = -\frac{\omega_p}{2Q_P} \left(1 \pm j\sqrt{4Q_P^2 - 1} \right)$$

Distance from origin in s-plane:

$$\begin{aligned} d^2 &= \left(\frac{\omega_p}{2Q_P} \right)^2 (1 + 4Q_P^2 - 1) \\ &= \omega_p^2 \end{aligned}$$



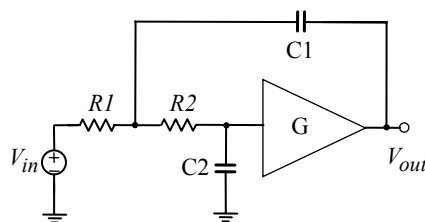
s-Plane



Implementation of Biquads

- Passive RC: only *real poles* → can't implement *complex conjugate poles*
- Terminated LC
 - Low power, since it is passive
 - Only fundamental noise sources → load and source resistance
 - As previously analyzed, not feasible in the monolithic form for $f < 350\text{MHz}$
- Active Biquads
 - Many topologies can be found in filter textbooks!
 - Widely used topologies:
 - Single-opamp biquad: *Sallen-Key*
 - Multi-opamp biquad: *Tow-Thomas*
 - Integrator based biquads

Active Biquad Sallen-Key Low-Pass Filter



$$H(s) = \frac{G}{1 + \frac{s}{\omega_p Q_p} + \frac{s^2}{\omega_p^2}}$$

$$\omega_p = \frac{1}{\sqrt{R_1 C_1 R_2 C_2}}$$

$$Q_p = \frac{\omega_p}{\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} + \frac{1-G}{R_2 C_2}}$$

- Single gain element
- Can be implemented both in discrete & monolithic form
- “Parasitic sensitive”
- Versions for LPF, HPF, BP, ...
 - Advantage: Only one opamp used
 - Disadvantage: Sensitive to parasitic – all pole no zeros

Addition of Imaginary Axis Zeros

- Sharpen transition band
- Can “notch out” interference
- High-pass filter (HPF)
- Band-reject filter

$$H(s) = K \frac{1 + \left(\frac{s}{\omega_Z}\right)^2}{1 + \frac{s}{\omega_P Q_P} + \left(\frac{s}{\omega_P}\right)^2}$$

$$|H(j\omega)|_{\omega \rightarrow \infty} = K \left(\frac{\omega_P}{\omega_Z}\right)^2$$

Note: Always represent transfer functions as a product of a gain term, poles, and zeros (pairs if complex). Then all coefficients have a physical meaning, and readily identifiable units.

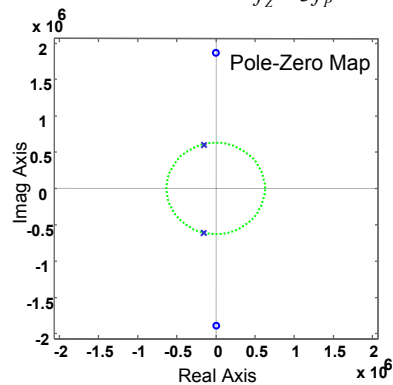
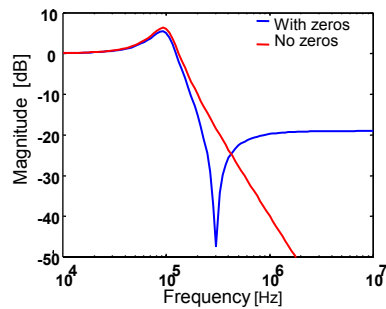
Imaginary Zeros

- Zeros substantially sharpen transition band
- At the expense of reduced stop-band attenuation at high frequencies

$$f_p = 100\text{kHz}$$

$$Q_p = 2$$

$$f_z = 3f_p$$

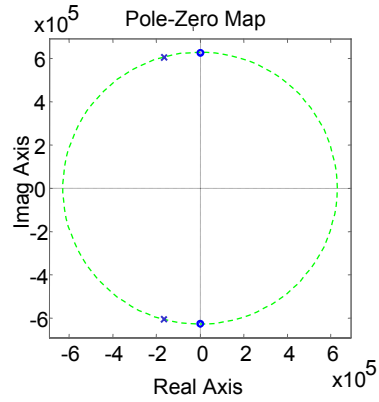
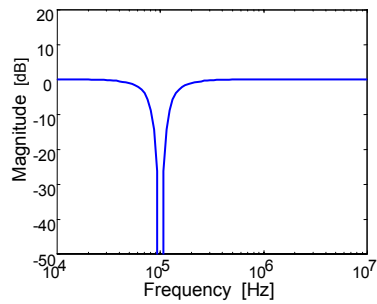


Moving the Zeros

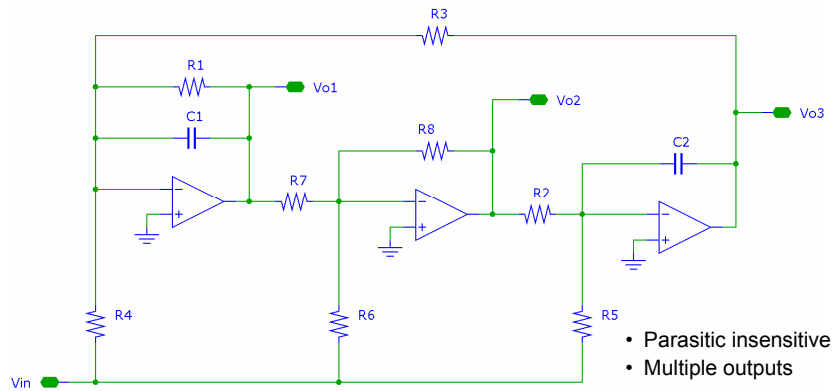
$$f_p = 100\text{kHz}$$

$$Q_p = 2$$

$$f_z = f_p$$



Tow-Thomas Active Biquad



Ref: P. E. Fleischer and J. Tow, "Design Formulas for biquad active filters using three operational amplifiers," Proc. IEEE, vol. 61, pp. 662-3, May 1973.

Frequency Response

$$\frac{V_{o1}}{V_{in}} = -k_2 \frac{(b_2 a_1 - b_1)s + (b_2 a_0 - b_0)}{s^2 + a_1 s + a_0}$$

$$\frac{V_{o2}}{V_{in}} = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}$$

$$\frac{V_{o3}}{V_{in}} = -\frac{1}{k_1 \sqrt{a_0}} \frac{(b_0 - b_2 a_0)s + (a_1 b_0 - a_0 b_1)}{s^2 + a_1 s + a_0}$$

- V_{o2} implements a general biquad section with arbitrary poles and zeros
- V_{o1} and V_{o3} realize the same poles but are limited to at most one finite zero

Component Values

$$b_0 = \frac{R_8}{R_3 R_5 R_7 C_1 C_2}$$

$$b_1 = \frac{1}{R_1 C_1} \left(\frac{R_8}{R_6} - \frac{R_1 R_8}{R_4 R_7} \right)$$

$$b_2 = \frac{R_8}{R_6}$$

$$a_0 = \frac{R_8}{R_2 R_3 R_7 C_1 C_2}$$

$$a_1 = \frac{1}{R_1 C_1}$$

$$k_1 = \sqrt{\frac{R_2 R_8 C_2}{R_3 R_7 C_1}}$$

$$k_2 = \frac{R_7}{R_8}$$

given a_i, b_i, k_i, C_1, C_2 and R_8

$$R_1 = \frac{1}{a_1 C_1}$$

$$R_2 = \frac{k_1}{\sqrt{a_0 C_2}}$$

$$R_3 = \frac{1}{k_1 k_2} \frac{1}{\sqrt{a_0 C_1}}$$

$$R_4 = \frac{1}{k_2} \frac{1}{a_1 b_2 - b_1} \frac{1}{C_1}$$

$$R_5 = \frac{k_1 \sqrt{a_0}}{b_0 C_2}$$

$$R_6 = \frac{R_8}{b_2}$$

$$R_7 = k_2 R_8$$

it follows that

$$\omega_p = \sqrt{\frac{R_8}{R_2 R_3 R_7 C_1 C_2}}$$

$$Q_p = \omega_p R_1 C_1$$