UC Berkeley

EE C245 – ME C218
Introduction to MEMS Design
Fall 2012

Prof. Clark T.-C. Nguyen

Dept. of Electrical Engineering & Computer Sciences
University of California at Berkeley
Berkeley, CA 94720

Lecture Module 8: Microstructural Elements

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   1

UC Berkeley

Outline

- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   2
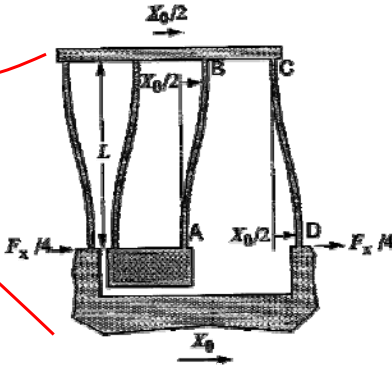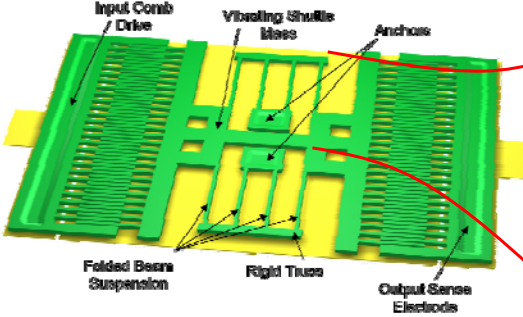
UC Berkeley

Bending of Beams

UC Berkeley
Beams: The Springs of Most MEMS

- Springs and suspensions very common in MEMS
  - ↳ Coils are popular in the macro-world; but not easy to make in the micro-world
  - ↳ Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS

Comb-Driven Folded Beam Actuator

Bending a Cantilever Beam

Free end condition

Clamped end condition:
At x=0:
  y=0
  dy/dx = 0

F

x' L

→ x

- **Objective**: Find relation between tip deflection $y(x=L_c)$ and applied load F
- **Assumptions**:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

Reaction Forces and Moments

Reaction } $M_R = M_1$
Moment

M_R

Point Load F

M_1 M_2

0   x L

Reaction Force → F_R
$F_R = F$

split

Moment due to F, hee:
$M_1 = FL$

Moment due to F, here:
$M_2 = F(L-x)$

M_R

0   x

F_R

$M_{x,l}$ $M_{x,r}$

$V_{x,l}$ $V_{x,r}$

M_3

F

x L

$M_3 = F(L-x)$

Reactions
(Senturia gives expressions)

For equilibrium: $M_{x,r} = M_3 = F(L-x)$
$V_{x,r} = F$

3

Sign Conventions for Moments & Shear Forces

Beam Segment in Pure Bending

Beam Segment in Pure Bending (cont.)

Thus, the axial strain @ z:

dx = Original (unstressed) segment length

$\epsilon_x = \dfrac{dL - dx}{dx} = -\dfrac{z}{R} \Rightarrow \boxed{\epsilon_x = -\dfrac{z}{R}}$

Thus, strain varies linearly along beam thickness, and has a maximum value

$\epsilon_{x,max} = \dfrac{h/2}{R}$

Of course, there is a corresponding axial stress:

$\sigma_x = \epsilon_x E = \boxed{-\dfrac{zE}{R} = \sigma_x}$

$-h/2$

$-\epsilon_{x,max}$ $\epsilon_{x,max}$ ϵ_x

$h/2$
z

$-h/2$ $\sigma_x = (+) \rightarrow$ tension

σ_x

$+h/2$ z $\sigma_x = (-) \rightarrow$ compression

This gradient in stress then generates a bending moment... in response to the applied moment!

Internal Bending Moment

Moment around this point

Small section of a beam bent in response to a transverse load

— Tension   Neutral Axis

M_O M_O
R $d\theta$

(+) radius of curvature

— Compression

Effectively, z = distance f/ the moment reference pt.

To get the bending moment:
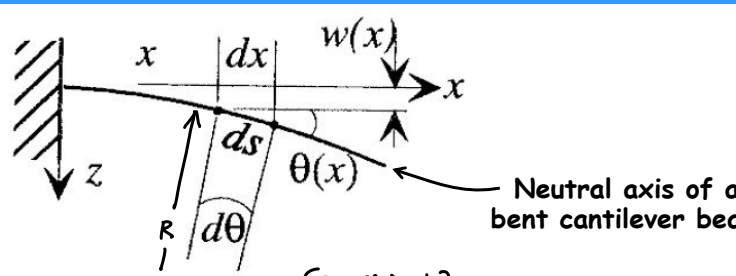$\Rightarrow$ integrate stress through the thickness of the beam

$M = \displaystyle\int_{-h/2}^{h/2} \left[(Wdz)\sigma_x \right] \cdot z = -\int_{-h/2}^{h/2} \dfrac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\dfrac{1}{12}Wh^3\right)\dfrac{E}{R}}$

force $\left[\sigma_x = -\dfrac{zE}{R} \right]$

$\dfrac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$\boxed{\dfrac{1}{R} = -\dfrac{M}{EI}}$

Note: (+) radius of curvature $\rightarrow$ (-) internal bending moment!

Differential Beam Bending Equation

UC Berkeley

Neutral axis of a bent cantilever beam

Write out geometric relationships:

$\cos\theta = \dfrac{dx}{ds} \rightarrow ds = \dfrac{dx}{\cos\theta}$ $\boxed{\text{Small Angle Approx.}}$ $\longrightarrow$ $ds \approx dx$

$\tan\theta = \dfrac{dw}{dx} =$ slope of beam @ any point $\longrightarrow$ $\theta \approx \dfrac{dw}{dx}$ (1)

$ds = Rd\theta \rightarrow \dfrac{1}{R} = \dfrac{d\theta}{ds}$ $\longrightarrow$ $\dfrac{1}{R} = \dfrac{d\theta}{dx}$ (2)

Inserting (1) in (2): $\dfrac{1}{R} = \boxed{\dfrac{d^2w}{dx^2} = -\dfrac{M}{EI}}$ $\boxed{\begin{array}{l}\text{Differential Equation for}\\ \text{Small Angle Bending of Beams}\end{array}}$

UC Berkeley

Example: Cantilever Beam w/ a Concentrated Load

Cantilever Beam w/ a Concentrated Load

Free end condition

F ← Point Load

Clamped end condition:
At x=0:
  w=0
 dw/dx = 0

M

h

x

L

Internal Moment @ position x: $M = -F(L-x)$

Thus: $\dfrac{d^2w}{dx^2} = \dfrac{F}{EI}(L-x)$, w/ $\begin{cases} \text{Clamped End B.C.'s: } W(x=0)=0, \dfrac{dW}{dx}(x=0)=0 \\ \text{Free End B.C.'s: none} \end{cases}$

Solve to get expression for w:
 ⇒ use Laplace; or use trial solution $w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$w = \dfrac{FL}{2EI}x^2\left(1-\dfrac{x}{3L}\right)$$ $\begin{bmatrix}\text{Deflection @ }x\text{ due to a point load} \\ F \text{ applied at } x=L\end{bmatrix}$

Cantilever Beam w/ a Concentrated Load

Free end condition

F ← Point Load

Clamped end condition:
At x=0:
  w=0
 dw/dx = 0

M

h

x

L

Maximum deflection @ $x=L$:

$$w_{max} = \left(\dfrac{L^3}{3EI}\right)F \rightarrow F = \left(\dfrac{3EI}{L^3}\right)W(x=L) = k_c W(x=L)$$

→ Note that in general, stiffness is a function of location x.

Where $k_c = \dfrac{3EI}{L^3} \triangleq$ stiffness @ location $x=L$

$\left[I = \dfrac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \dfrac{1}{4}EW\dfrac{h^3}{L^3}}$

Ex. $L=100\mu m$, $W=2\mu m$, $h=2\mu m$
polysilicon → $E = 150\,GPa$

$k_c = \dfrac{1}{4}(150G)(2\mu)\left(\dfrac{2\mu}{100\mu}\right)^3 = \underline{0.6\ N/m}$

Maximum Stress in a Bent Cantilever

UC Berkeley

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x) \implies \text{maximized where } R \to 0$$

↳ occurs at the support, where $x=0$:

$$[x=0] \implies \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$

slope = 0

Strain is maximized:

① At top surface → tensile
② At bottom surface → compressive

$$\epsilon_{max} = -\frac{z}{R} = \frac{h}{2}\frac{1}{R} = \frac{h}{2}\frac{FL}{EI}$$

$$\left[I = \frac{1}{12}Wh^3\right] \implies \epsilon_{max} = \frac{h}{2}\frac{FL}{E}\frac{12}{Wh^3 \cdot 2} = \frac{6L}{EWh^2}F$$

$$\therefore \boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F} \quad \left(\begin{array}{c}\text{Maximum Stress in a}\\ \text{Bent Cantilever}\end{array}\right)$$

UC Berkeley
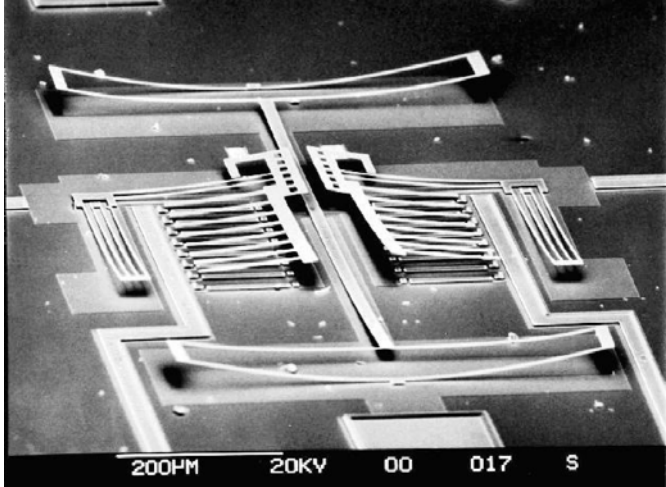
Stress Gradients in Cantilevers

Vertical Stress Gradients

- Variation of residual stress in the direction of film growth
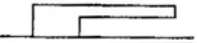- Can warp released structures in z-direction

Stress Gradients in Cantilevers

- **Below**: surface micromachined cantilever deposited at a high temperature then cooled → assume compressive stress
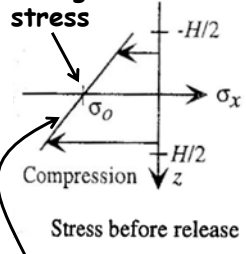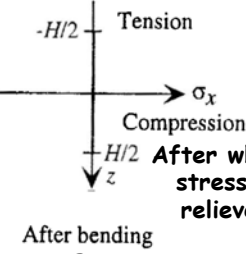

Average stress

Stress gradient

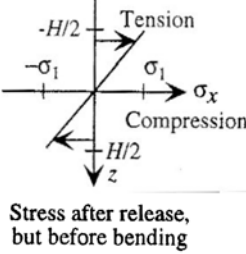Once released, beam length increases slightly to relieve average stress

But stress gradient remains → induces moment that bends beam

After which, stress is relieved

Stress Gradients in Cantilevers (cont)

Find the radius of curvature.

Prior to release, axial stress is: $\sigma = \sigma_0 - \dfrac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-H/2}^{H/2} \left[(W \cdot dz) \, \sigma \right] \cdot z = W \int_{-H/2}^{H/2} \left(z \sigma_0 - \frac{\sigma_1 z^2}{(H/2)} \right) dz = W \left(\frac{1}{2} \sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H} \right) \Bigg|_{-H/2}^{H/2}$$

$$= W \left(\frac{1}{2} \sigma_0 \frac{H^2}{4} - \frac{2}{3} \sigma_1 \frac{H^2}{8} - \frac{1}{2} \sigma_0 \frac{H^2}{4} - \frac{2\sigma_1 H^2}{3(8)} \right) = -\frac{1}{6} \sigma_1 W H^2 = M_x$$

Thus, the radius curvature is:

$$\frac{1}{R} = -\frac{M_x}{E'I} \longrightarrow R = -\frac{E'I}{M_x} = -\frac{E'\left(\frac{1}{12}WH^3\right)}{-\frac{1}{6}\sigma_1 WH^2} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

Biaxial Stress Gradient

$\left[I = \frac{1}{12}Wh^3 \right]$

$$\Rightarrow \boxed{R = \frac{1}{2}\frac{E}{(1-\gamma)}\frac{H}{\sigma_1}} \quad \left[\begin{array}{c}\text{Radius of Curvature}\\\text{for a Cantilever}\\\text{w/ Stress Gradient}\end{array}\right]$$

$$\boxed{\sigma_1 = \frac{1}{2}\frac{E}{(1-\gamma)}\frac{H}{R}} \Rightarrow R \text{ can be used to determine stress gradient}$$

Measurement of Stress Gradient

- **Use cantilever beams**
  - ↳ **Strain gradient (Γ = slope of strain-thickness curve) causes beams to deflect up or down**
  - ↳ **Assuming linear strain gradient Γ, $z = \Gamma L^2/2$**

compressive

tensile

[P. Krulevitch Ph.D.]

−

+

UC Berkeley

Folded-Flexure Suspensions

UC Berkeley

Folded-Beam Suspension

- Use of folded-beam suspension brings many benefits
  - Stress relief: folding truss is free to move in y-direction, so beams can expand and contract more readily to relieve stress
  - High y-axis to x-axis stiffness ratio

Folding Truss

Comb-Driven Folded Beam Actuator

Beam End Conditions

TABLE 4.1
Types of commonly used support conditions for beams and frames

Type of support	Displacement boundary conditions	Force boundary conditions
FREE	None	All, as specified
PINNED	$u = 0$ $w = 0$	Moment is specified
ROLLER (vertical)	$u = 0$	Transverse force and moment are specified
ROLLER (horizontal)	$w = 0$	Horizontal force and bending moment are specified
FIXED or CLAMPED	$u = 0$ $w = 0$ $dw/dx = 0$	None specified

[From Reddy, Finite Element Method]

Common Loading & Boundary Conditions

- Displacement equations derived for various beams with concentrated load F or distributed load f
- Gary Fedder Ph.D. Thesis, EECS, UC Berkeley, 1994

	cantilever	guided-end	fixed-fixed
	$x = \frac{F_x L}{Ehw}$	$x = \frac{F_x L}{Ehw}$	$x = \frac{F_x L}{4Ehw}$
	$y = 4\frac{F_y}{Eh}\frac{L^3}{w^3}$	$y = \frac{F_y}{Eh}\frac{L^3}{w^3}$	$y = \frac{1}{16}\frac{F_y}{Eh}\frac{L^3}{w^3}$
	$z = 4\frac{F_z}{Ew}\frac{L^3}{h^3}$	$z = \frac{F_z}{Ew}\frac{L^3}{h^3}$	$z = \frac{1}{16}\frac{F_z}{Ew}\frac{L^3}{h^3}$

(a) Concentrated load.

	cantilever	guided-end	fixed-fixed
	$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{4E}$
	$y = \frac{3}{2}\frac{f_y}{Eh}\frac{L^4}{w^3}$	$y = \frac{1}{2}\frac{f_y}{Eh}\frac{L^4}{w^3}$	$y = \frac{1}{32}\frac{f_y}{Eh}\frac{L^4}{w^3}$
	$z = \frac{3}{2}\frac{f_z}{Ew}\frac{L^4}{h^3}$	$z = \frac{1}{2}\frac{f_z}{Ew}\frac{L^4}{h^3}$	$z = \frac{1}{32}\frac{f_z}{Ew}\frac{L^4}{h^3}$

(b) Distributed load.

Series Combinations of Springs

UC Berkeley

- **For springs in series w/ one load**
  - ✎ Deflections add
  - ✎ Spring constants combine like "resistors in parallel"

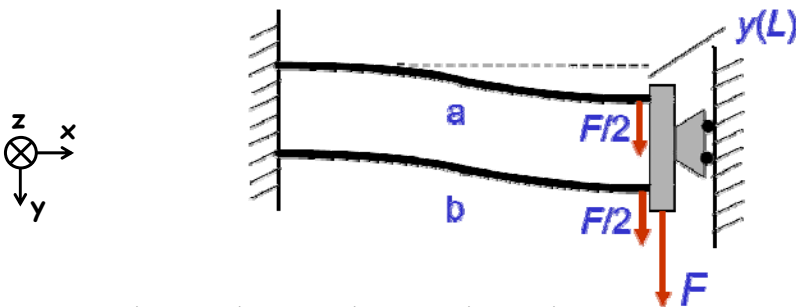$$Y(L) = F/k = 2\, y(L_c) = 2\, (F/k_c) = F(1/k_c + 1/k_c)$$

Compliances effectively add:

$$1/k = 1/k_c + 1/k_c \quad \rightarrow \quad k = k_c || k_c$$

Parallel Combinations of Springs

UC Berkeley

- **For springs in parallel w/ one load**
  - ✎ Load is shared between the two springs
  - ✎ Spring constant is the sum of the individual spring constants

$$Y(L) = F/k = F_a/k_a = F_b/k_b = (F/2)\,(1/k_a)$$

$$k = 2\, k_a$$

Folded-Flexure Suspension Variants

• **Below**: just a subset of the different versions
• All can be analyzed in a similar fashion

(a) Inner fold, continuous truss   (b) Inner fold, discontinuous truss   (c) Outer fold, continuous truss   (d) Outer fold, discontinuous truss

[From Michael Judy, Ph.D. Thesis, EECS, UC Berkeley, 1994]

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   27

Deflection of Folded Flexures

This equivalent to two cantilevers of length $L_c = L/2$

Composite cantilever free ends attach here

Half of F absorbed in other half (symmetrical)

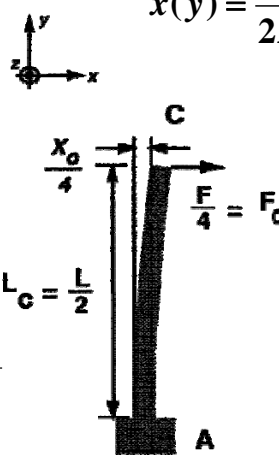4 sets of these pairs, each of which gets ¼ of the total force F

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   28

Constituent Cantilever Spring Constant

From our previous analysis:

$$x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right) = \frac{F_c y^2}{6EI_z}(3L_c - y)$$

From which the spring constant is:
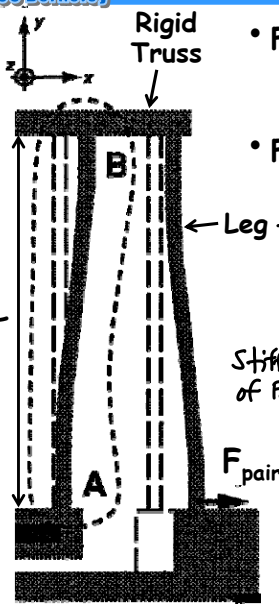
$$k_c = \frac{F_c}{x(L_c)} = \frac{3EI_z}{L_c^3}$$

Inserting $L_c = L/2$

$$k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}$$

$\frac{x_o}{4}$

$\frac{F}{4} = F_c$

$L_c = \frac{L}{2}$

C

A

Overall Spring Constant

Rigid Truss

*Four pairs of clamped-guided beams
   ↳ In each pair, beams bend in series
   ↳ (Assume trusses are inflexible)*
Force is shared by each pair → $F_{pair} = F/4$

total force on shuttle

B

←Leg→ Displacement of two legs add
   ↳ thus, springs are in series:

Stiffness of Pair

$$x = \frac{F_{pair}}{k_{pair}} = \frac{F_{pair}}{(k_{leg} || k_{leg})} = \left(\frac{F}{4}\right)\left(\frac{1}{k_{leg}} + \frac{1}{k_{leg}}\right)$$

From before: $k_{leg} = k_c || k_c = \frac{k_c}{2}$
Thus:
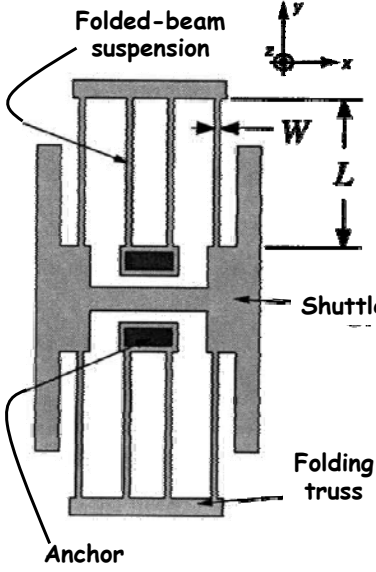
$$x = \left(\frac{F}{4}\right)\left(\frac{2}{k_c} + \frac{2}{k_c}\right) = \frac{F}{k_c} = \frac{F}{k_{tot}}$$

$$\Rightarrow \boxed{k_{tot} = k_c = \frac{24EI_z}{L^3}}$$

L

A

F_{pair}

Folded-Beam Stiffness Ratios

Folded-beam suspension

W

L

Shuttle

Folding truss

Anchor

- In the x-direction:

$$k_x = \frac{24EI_z}{L^3}$$

- In the z-direction:
  ☞ Same flexure and boundary conditions

$$k_z = \frac{24EI_x}{L^3}$$

- In the y-direction:

[See Senturia, §9.2] $k_y = \dfrac{8EWh}{L}$

- Thus: $\boxed{\dfrac{k_y}{k_x} = 4\left(\dfrac{L}{W}\right)^2}$ **Much stiffer in y-direction!**

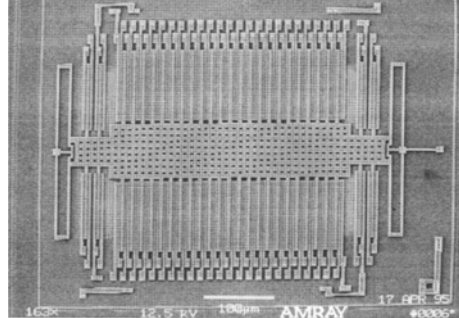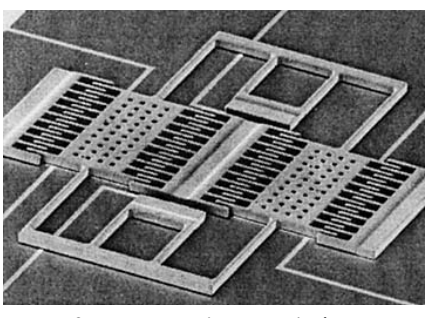EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   31
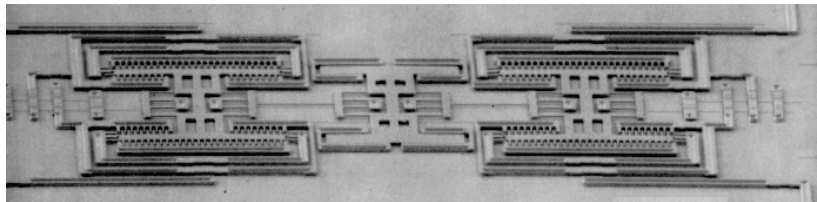
Folded-Beam Suspensions Permeate MEMS

Accelerometer [ADXL-05, Analog Devices]

Gyroscope [Draper Labs.]

Micromechanical Filter [K. Wang, Univ. of Michigan]

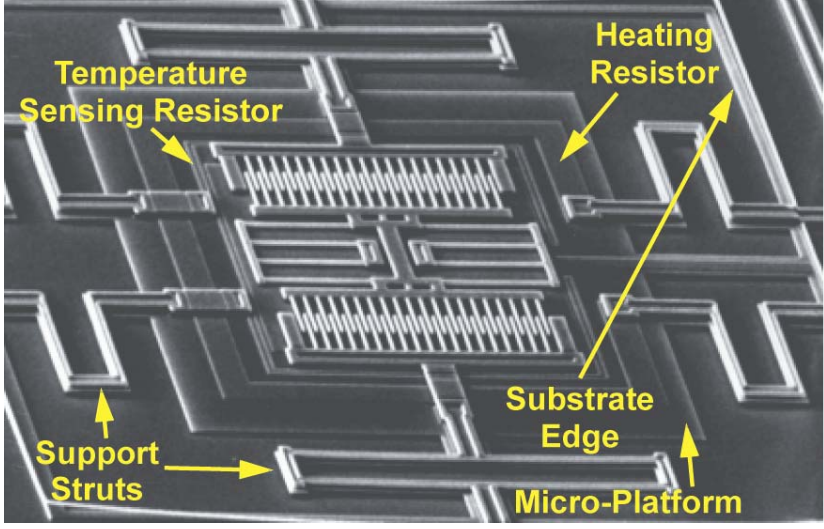EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   32

Folded-Beam Suspensions Permeate MEMS

UC Berkeley

• **Below**: Micro-Oven Controlled Folded-Beam Resonator

Temperature Sensing Resistor

Heating Resistor

Support Struts

Substrate Edge

Micro-Platform

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   33

UC Berkeley

Stressed Folded-Flexures

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   34

Clamped-Guided Beam Under Axial Load

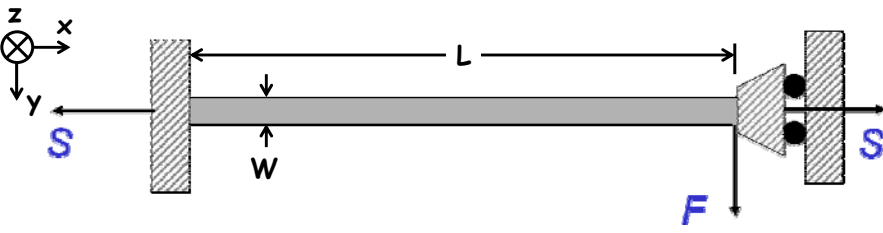- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
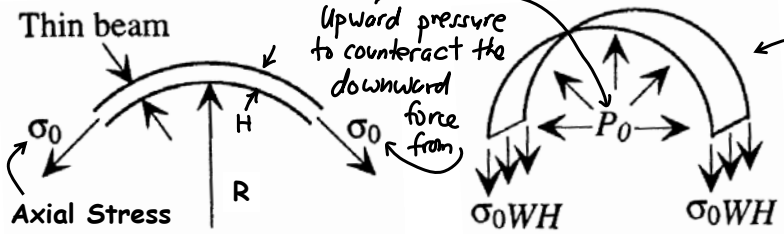- Consider small deflection case: y(x) ≪ L

Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F\delta(x-L)$$

Axial Load   Unit impulse @ x=L

The Euler Beam Equation

Thin beam

Upward pressure to counteract the downward force from

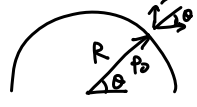σ_0 Axial Stress   H   R σ_0 P_0 $\sigma_0 WH$ $\sigma_0 WH$

- Axial stresses produce no net horizontal force; but as soon as the beam is bent, there is a net downward force
  - For equilibrium, must postulate some kind of upward load on the beam to counteract the axial stress-derived force
  - For ease of analysis, assume the beam is bent to angle π

Downward Vertical Force = $2\sigma_0 WH$

Upward Force due to P_0:

$P_z(\theta) = P_0 \sin\theta$

$F_u = \int_0^\pi (P_0 \sin\theta) W (Rd\theta)$

$= -P_0 WR\cos\theta \Big|_0^\pi = 2RWP_0$

The Euler Beam Equation

[Equilibrium] $\Rightarrow$ $2RWP_0 = 2\sigma_0 WH \longrightarrow P_0 = \frac{\sigma_0 H}{R}$

$\left[q_0 = \frac{beam\ load}{unit\ length} = P_0 W,\ \frac{1}{R} = \frac{d^2 w}{dx^2} \right] \Rightarrow q_0 = \sigma_0 WH \frac{d^2 w}{dx^2}$

beam displacement

Using the differential beam bending equation:

$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI} \leftarrow \frac{load}{unit\ length}$

$EI \frac{d^4 w}{dx^4} = q + q_0 \leftarrow$ external load

equiv. load accounting for the axial stress contribution to the bending stiffness

$\left[q_0 = \sigma_0 WH \frac{d^2 w}{dx^2} \right] \Rightarrow EI \frac{d^4 w}{dx^4} - (\sigma_0 WH) \frac{d^2 w}{dx^2} = q$ [Euler Beam Equation]

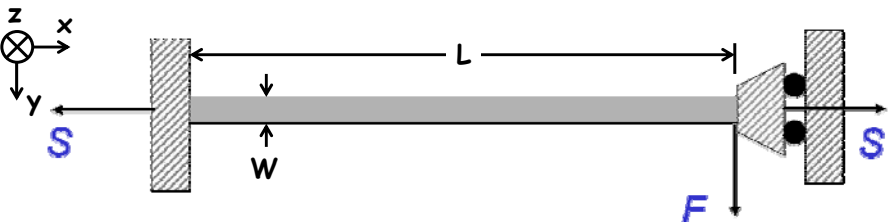tension in the beam = S $\leftarrow$ a force

Note: Use of the full bend angle of π to establish conditions for load balance; but this returns us to case of small displacements and small angles

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   37

Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L

Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F\delta(x-L)$$

Axial Load   Unit impulse @ x=L

EE C245: Introduction to MEMS Design   LecM 8   C. Nguyen   9/28/07   38

Solving the ODE

- Can solve the ODE using standard methods
  - ↳ Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - ↳ For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

$S > 0$ (tension)

$$k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

$S < 0$ (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

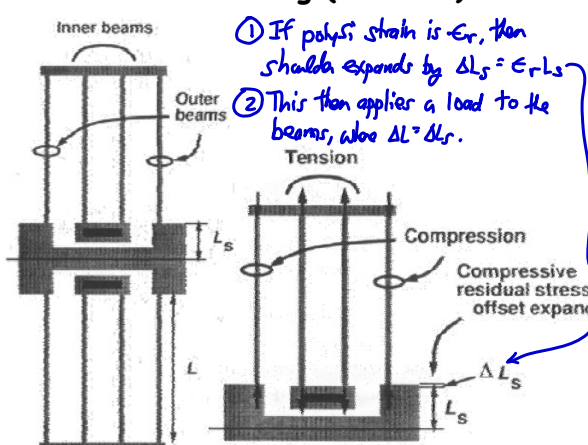where $p = \sqrt{\dfrac{|S|}{EI_z}}$

Design Implications

- Straight flexures
  - ↳ Large tensile S means flexure behaves like a tensioned wire (for which $k^{-1} = L/S$)
  - ↳ Large compressive S can lead to buckling ($k^{-1} \rightarrow \infty$)

- Folded flexures
  - ↳ Residual stress only partially released
  - ↳ Length from truss to shuttle's centerline differs by L_s for inner and outer legs

Inner beams

Outer beams

Tension

Compression

Compressive residual stress: offset expands

L_s

L

L_s

ΔL_s

① If polySi strain is ϵ_r, then shuttle expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, where $\Delta L = \Delta L_s$.

③ Beam strain:
$$\epsilon_b = \frac{\Delta L}{L} = \frac{\Delta L_s}{L} = \epsilon_r \frac{L_s}{L}$$

over ↻

Effect on Spring Constant

UC Berkeley

- Residual compression on outer legs with same magnitude of tension on inner legs:

  Strain in the polySi

  Beam Strain: $\varepsilon_b = \pm \varepsilon_r \left(\dfrac{L_s}{L} \right)$; Stress Force: $S = \pm E \varepsilon_r \left(\dfrac{L_s}{L} \right) Wh$

  Strain in the beams

  Expansion of the Shoulder = ΔL_s
  = $\varepsilon_r L_s$ ← This expansion applies a local on the beams

- Spring constant becomes:

  $\varepsilon_b = \dfrac{\Delta L}{L} = \dfrac{\varepsilon_r L_s}{L}$

  $[\Delta L = \Delta L_s]$

  $\varepsilon_b = \varepsilon_r \dfrac{L_s}{L}$

  $$k = 4(k_{com}^{-1} + k_{ten}^{-1})^{-1}$$

  $$k = 4 \left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|} \right]^{-1}$$

  of the flexure

- <u>Remedies</u>:
  - ✎ Reduce the shoulder width L_s to minimize stress in legs
  - ✎ Compliance in the truss lowers the axial compression and tension and reduces its effect on the spring constant