Lecture 20: Equivalent Circuits
- **Announcements:**
- HW#6 online
- Modules 11 and 12 online
- Project handed out and described last time
- Graded midterm passed out last time
- --
- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - ↳ Lumped Mechanical Equivalent Circuits
  - ↳ Electromechanical Analogies
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - ↳ Energy Conserving Transducers
  - ↳ Parallel-Plate Capacitive Transducers
- --
- **Last Time:**
- Derived the following for the resonance frequency of a folded beam resonator:

$$\omega_0 = \sqrt{\frac{k_c}{M_{eq}}} \quad , \text{ where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

$$\left[\text{Resonance Freq. of a Folded-Beam Suspended Shuttle} \right]$$

- Looked briefly at Module 10 slides 21–31, but very quickly – you should go through it again on your own

---

Equivalent Dynamic Mass

Location on the Folding Truss
→ F_2 *

Location on the Shuttle

Equivalent Mass:

$$\text{Equiv. Mass} = M_{eqx} = \frac{KE_{max}}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2}\rho A \int_0^\ell V^2(x)\,dx}{\frac{1}{2}V_x^2}$$

velocity
velocity @ location x

$$M_{eq(shuttle)} = \frac{KE_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\omega_0^2 X_0^2 \left(\frac{1}{2}\right)\left(M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right)}{\frac{1}{2}\omega_0^2 X_0^2}$$

$$M_{eq(shuttle)} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b \quad \Leftarrow \text{Dynamic Mass}$$

static mass = $\rho \times$ (volume)

$$* \quad M_{eq(truss)} = \frac{\omega_o^2 x_o^2 \left(\frac{1}{4}\right) \left[M_s + \frac{1}{4}M_t + \frac{13}{35}M_b\right]}{\frac{1}{2}\left(\frac{1}{4}\right)\omega_o^2 x_o^2}$$

$$M_{eq(truss)} = 4\left[M_s + \frac{1}{4}M_t + \frac{13}{35}M_b\right]$$

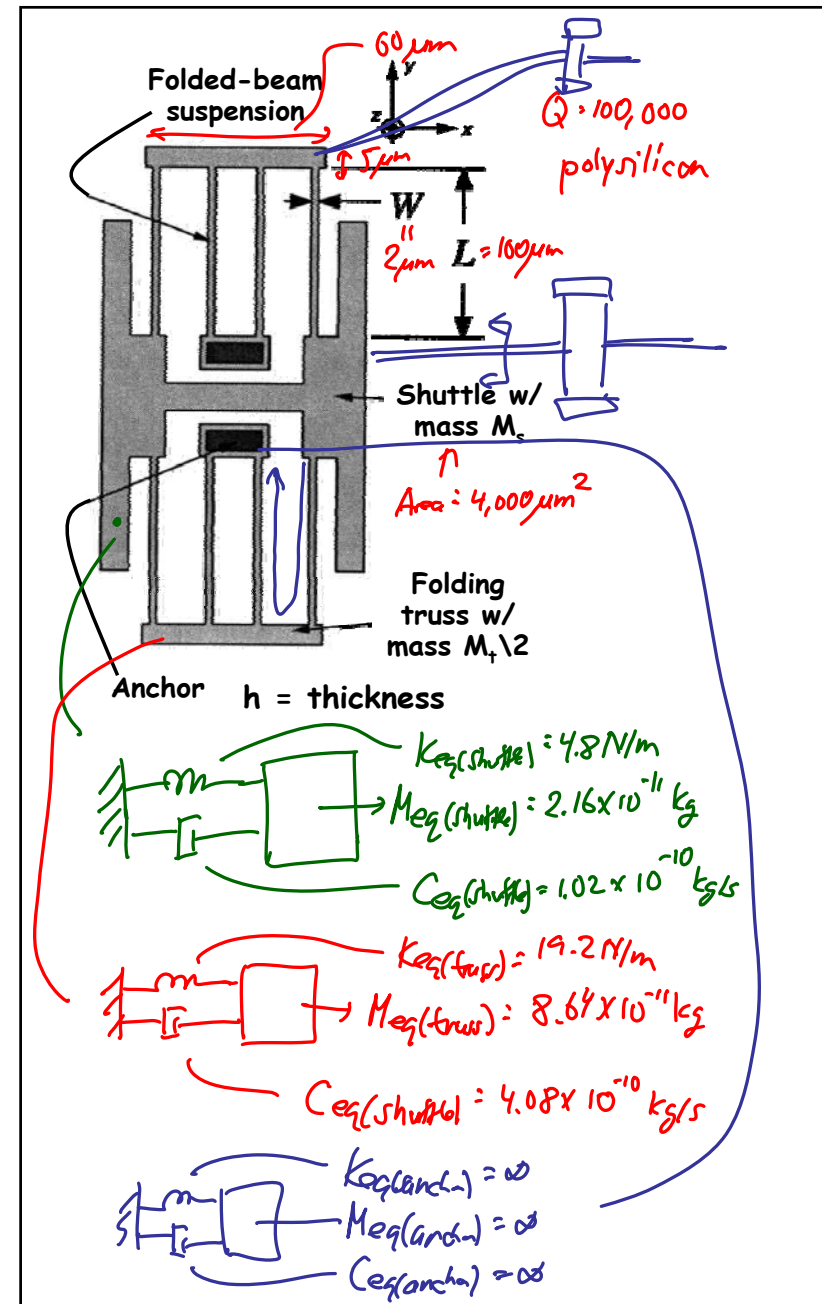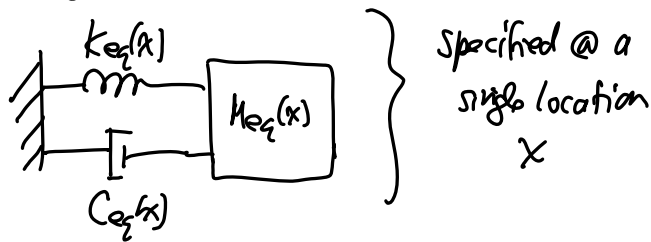Equiv. Dynamic Mass

Equiv. Dynamic Stiffness

$$\omega_o = \sqrt{\frac{K_{eq}(x)}{M_{eq}(x)}} \longrightarrow \boxed{K_{eq}(x) = \omega_o^2 M_{eq}(x)}$$

$\Rightarrow$ large equiv. mass & large equiv. stiffness go hand in hand

Equiv. Dynamic Damping

$$Q = \frac{\omega_o L}{R}$$

$$Q = \frac{\omega_o M_{eq}(x)}{C_{eq}(x)} \longrightarrow C_{eq}(x) = \frac{\omega_o M_{eq}(x)}{Q} = \frac{\sqrt{K_{eq}(x)M_{eq}(x)}}{Q}$$

damping

$K_{eq}(x)$
$M_{eq}(x)$
$C_{eq}(x)$

Specified @ a single location x

Folded-beam suspension

60 µm

$Q > 100,000$
polysilicon

W
2" L = 100µm

Shuttle w/ mass M_s

Area $= 4,000 \mu m^2$

Folding truss w/ mass $M_t\backslash 2$

Anchor   h = thickness

$K_{eq(shuttle)} = 4.8 N/m$
$M_{eq(shuttle)} = 2.16 \times 10^{-11} kg$
$C_{eq(shuttle)} = 1.02 \times 10^{-10} kg/s$

$K_{eq(truss)} = 19.2 N/m$
$M_{eq(truss)} = 8.64 \times 10^{-11} kg$
$C_{eq(shuttle)} = 4.08 \times 10^{-10} kg/s$

$K_{eq(anchor)} = \infty$
$M_{eq(anchor)} = \infty$
$C_{eq(anchor)} = \infty$

Electromechanical Analogies

$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$
(off resonance)

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

$\Rightarrow$ using phasor concept:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

Impedance looking in:

$$\frac{V}{i} = j\omega l_x + \frac{1}{j\omega c_x} + r_x$$

$$V = j\omega l_x i + \frac{(1/c_x)}{j\omega}i + r_x i \qquad i = \dot{q}$$

$\Rightarrow$ by analogy:

$$\boxed{\begin{array}{ll} F \rightarrow V & m_{eq} \rightarrow l_x \\ \dot{x} \rightarrow i & k_{eq} \rightarrow \frac{1}{c_x} \qquad c_{eq} \rightarrow r_x \end{array}}$$

(Parameter Relationships in the **Current Analogy**)

• Mechanical-to-electrical correspondence in the current analogy:

Mechanical Variable	Electrical Variable
Damping, c	Resistance, R
Stiffness^{-1}, k^{-1}	Capacitance, C
Mass, m	Inductance, L
Force, f	Voltage, V
Velocity, v	Current, I

Lowpass Biquad Transfer Function

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

$\Rightarrow$ convert to full phasor form:

$$F = (j\omega)(j\omega x)m_{eq} + \frac{k_{eq}}{j\omega}(j\omega x) + c_{eq}(j\omega x)$$

$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\omega^2\frac{m_{eq}}{k_{eq}} + 1 + j\frac{c_{eq}\omega}{k_{eq}}\right]^{-1} \quad \text{---} *$$

$$\left[\frac{k_{eq}}{m_{eq}} = \omega_0^2, \quad Q = \frac{m_{eq}\omega_0}{c_{eq}} = \frac{k_{eq}}{\omega_0 c_{eq}} \rightarrow \frac{k_{eq}}{c_{eq}} = Q\omega_0 \right]$$

$$* \quad \frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j\frac{\omega}{Q\omega_0} \right]^{-1}$$

$$\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{Q\omega_0}}$$

$$x = \frac{QF}{jk_{eq}}$$

90° phase shift

k_{eq}^{-1}

ω_0

· Go through Module 11, slides 11-22

Basic Physics of Electrostatic Actuation

+q F_e F_e -q

g

V

Goal: Determine the gap spacing g as a function of input variables.

Nob. Assume the plates are supported elastically.

1st: Determine the energy of the system.

2nd: Ask: what can I do to Δ the energy of the system?

① change the charge q
② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e\Delta g$$

$$\boxed{dW = V dq + F_e dg}$$