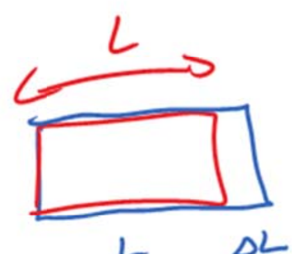
EE245 Discussion 10/18/10

Monday, October 18, 2010
11:20 AM

Today:

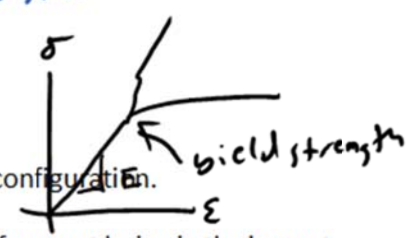
Definition of stress/strain
Hooke's law
Stress/strain gradients
Beam bending equations
Second moment of inertia
Example:
Bending profile for a cantilever
-angle, moment, shear, distributed load
Strategies for solving for beam deflections given boundary conditions
Beam combos/flexures

Stress
σ = Force/Area
Materials experience stresses when acted on by external forces
(+) stress is tensile, (-) stress is compressive.
If not acted on by an external force, **a material under tensile stress will contract.**
" " " " " " " ", **a material under compressive stress will expand.**
Units: Pa (Pascal) ~ [N/m^2]
Strain
ε = (Change in length) / (length)
$\varepsilon = \Delta L/L$
(+) strain is an expansion
(-) strain is a contraction
Units: Unitless ~ [parts per million, ppm]



$\varepsilon = \frac{\Delta L}{L}$

Hooke's Law
Relates stress to strain. Young's Modulus ~150 GPa for polysilicon.
$\sigma = E\,\varepsilon$
This is the more general form of the familiar
F = k x
Explanation:
Atoms or molecules in a solid arrange themselves in the lowest energy configuration.
==> They have an equilibrium spacing that they want to maintain.
Materials contract or expand (change their strain) to eliminate stresses from not being in the lowest energy configuration.



==> The spring force is electromagnetic in nature.

Stress/Strain gradients
A gradient is just a spacial derivative

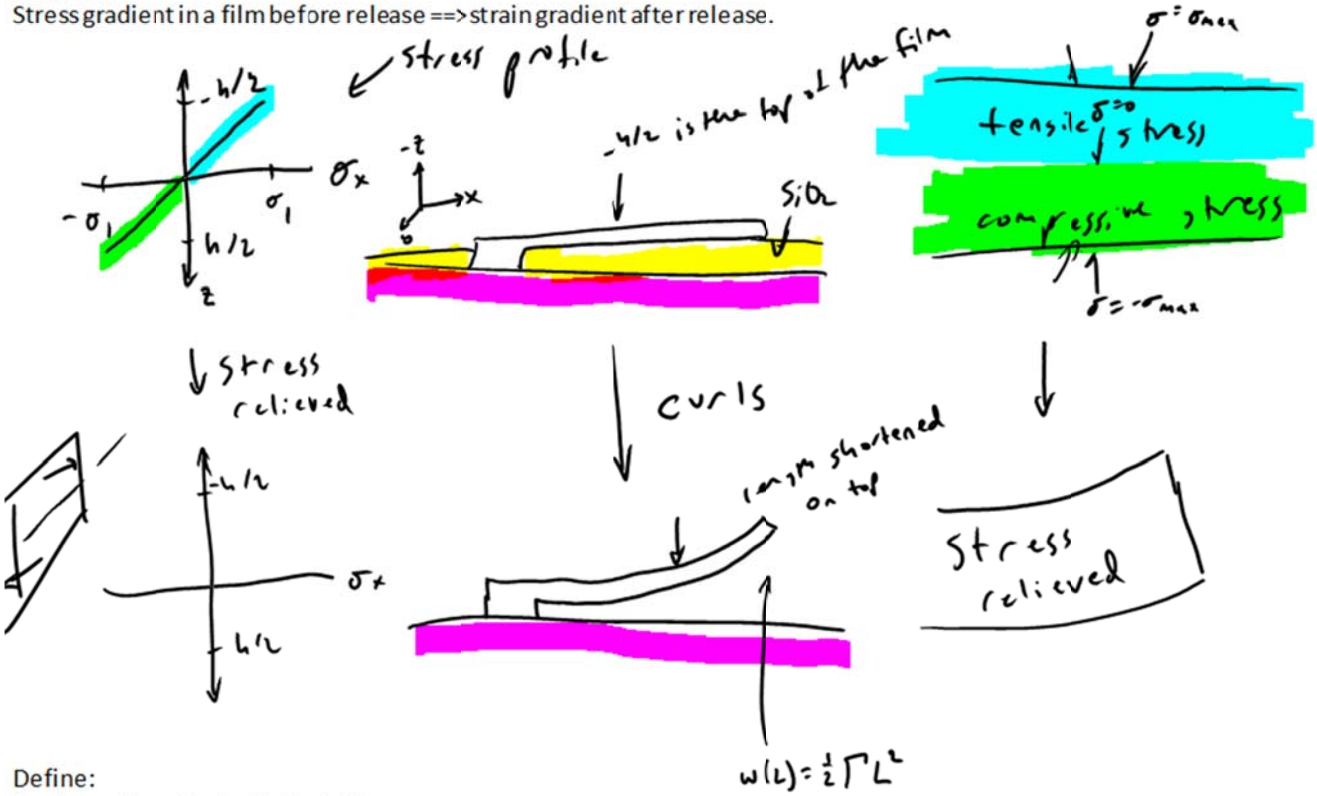
$$\frac{df}{d\hat{l}} = \nabla f \cdot l$$

In 3-D:
$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

In 1-D: the gradient is a scalar (number) describing how a quantity changes with respect to distance.
If the quantity changes linearly, the gradient is a constant.

==> Stress/strain gradient means the stress/strain is a function of position in the material.

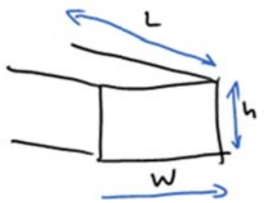
Example:
Stress gradient in a film before release ==> strain gradient after release.

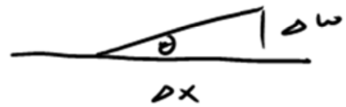


Define:
Strain gradient $\Gamma = d\varepsilon_x/dz$ (in 1-D)
$\Gamma = d\sigma_x/dz\,/E$
Units: [m^{-1}]

The stress gradient creates an internal bending moment...
$M = \int_{-h/2}^{h/2} \sigma_x(z) zW\,dz$, which is related to the second derivative of deflection through $\frac{d^2w}{dx^2} = -\frac{M}{EI}$ ...
Where I is the second moment of inertia (property of a cross section)... More on this below...

Beam bending





$\tan\theta = \theta = \frac{\Delta w}{\Delta x}$

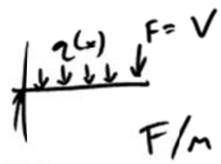
Notational definitions:

w W width

w(x) deflection

$\frac{\partial w}{\partial x} = \theta$

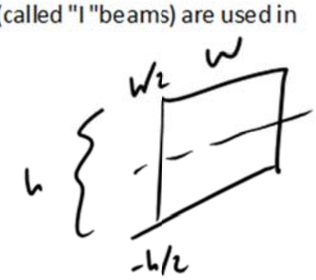
w(x)* (y, z, etc. are also used) is the **deflection**.
θ (φ is used too) is the **slope** of the beam
M is the bending **moment**
V (Q is also used) is the **shear** stress
q is the **distributed load**
ρ (r) is the **radius of curvature**
*don't confuse width w with deflection w....



F/m

Second moment of inertia
$I = \int y^2 dA$, where y is the perpendicular distance between the neutral axis and the area.
$I = \frac{1}{12} W h^3$, for a rectangular beam cross section.
I is a measure of a cross section's resistance to bending. This is why H-shaped beams (called "I "beams) are used in buildings... they have a high 2nd moment of inertia.

~[m⁴]



$I = \int_{-h/2}^{h/2} W y^2 dy$

$I = 2\left(W\frac{y^3}{3}\right)\Big|_{y=0}^{y=h/2}$

$\boxed{I = \frac{1}{12} W h^3}$

Simple beam bending theory
Assumes that deflections are small, angles are small, beams are very thin.

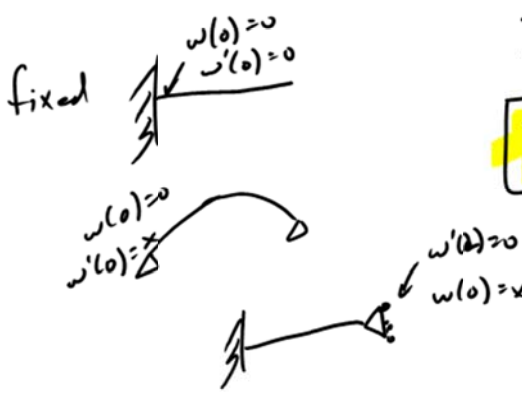
For an exact treatment of beam bending, use **Timoshenko beam bending theory.**
==> The math is more difficult.
e.g.

$$EI\frac{\partial^4 w}{\partial x^4} + N\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} - \left(J + \frac{mEI}{\kappa AG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{mJ}{\kappa AG}\frac{\partial^4 w}{\partial t^4} = q + \frac{J}{\kappa AG}\frac{\partial^2 q}{\partial t^2} - \frac{EI}{\kappa AG}\frac{\partial^2 q}{\partial x^2}$$

(source: wikipedia. Timoshenko beam theory)

Important equations:

$$EI\frac{d^4 w}{dx^4} = q(x)$$
$$EI\frac{d^3 w}{dx^3} = -V(x)$$
$$EI\frac{d^2 w}{dx^2} = -M(x)$$
$$\frac{dw}{dx} = \theta(x)$$
$$\frac{d^2 w}{dx^2} = 1/\rho$$



Simple facts:
If a constant distributed load q(x) is applied to a beam, the solution is fourth order.
==> need four boundary conditions

If a constant shear force (e.g. due to a point load) is applied to a beam (q(x)=0), the solution is third order.
==> three boundary conditions are required.

If only a constant bending moment is applied (e.g. from a strain gradient), the solution is second order.

Example:
Bending of a cantilever beam due to a point load.

Derive the deflection as a function of position

$\frac{\partial^4 w}{\partial x^4} = 0$ $\qquad$ $\frac{\partial^3 w}{\partial x^3} = -\frac{V}{EI}$ $\qquad$ BC $M = 0$ @ L

$I = \frac{1}{12} w h^3$

$$\int w''' \, dx = -\frac{Fx}{EI} + C_3 = \frac{\partial^2 w}{\partial x^2}$$

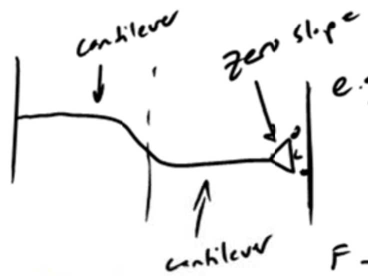
$$-\frac{FL}{EI} + C_3 = 0$$
$$\Rightarrow C_3 = \frac{FL}{EI}$$

$$\int -\frac{Fx}{EI} + \frac{FL}{EI} \, dx = \frac{\partial w}{\partial x}$$

BC: $w'(0) = 0$

$$-\frac{Fx^2}{2EI} + \frac{FLx}{EI} + C_2 \Rightarrow$$

$\Rightarrow C_2 = 0$

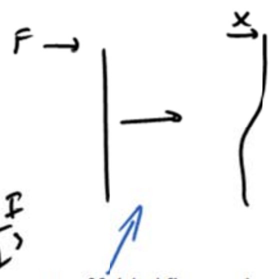


e.g. folded flexure beam

$w(0) = 0$

0

$$w(x) = -\frac{Fx^3}{6EI} + \frac{FLx^2}{2EI} + C_1$$



$$w(L) = -\frac{FL^3}{6EI} + \frac{FL^3}{2EI} = \frac{FL^3}{3EI}$$

$$F = kx \Rightarrow k = \frac{F}{x}$$

Useful results:

Cantilever stiffness:
$k_c = 1/4\, E\, w\, h^3/L_c^3$

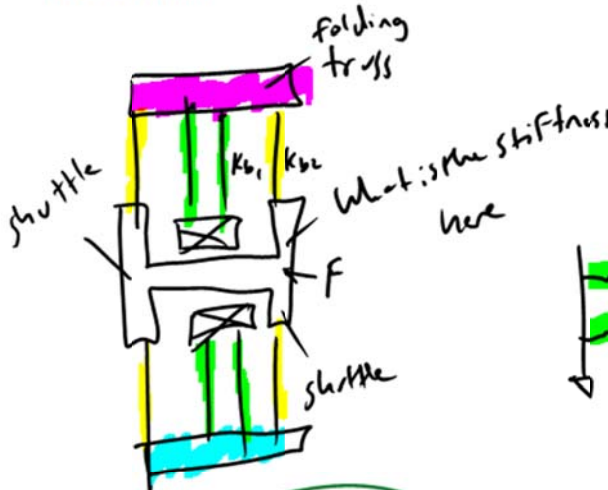
$k_c = \frac{Ewh^3}{4L_c^3}$ $\leftarrow$ $k = \frac{3EI}{L^3}$

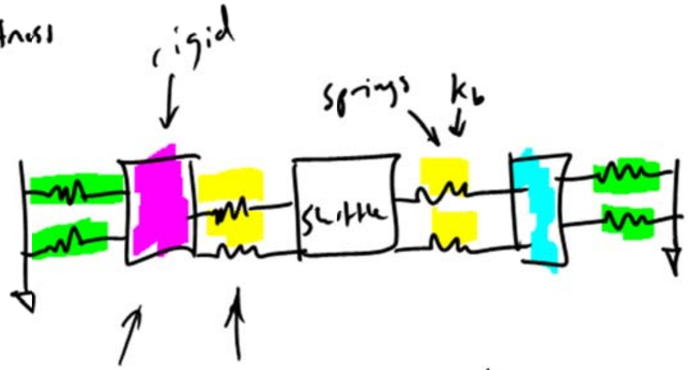
Fixed guided beam stiffness (elements of folded flexures)
FG beam = Two $L_b/2$ cantilevers in series...

$k_b = \frac{Ewh^3}{L_b^3}$

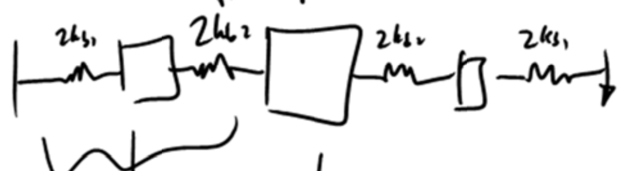
Beam combos

Folded flexure:

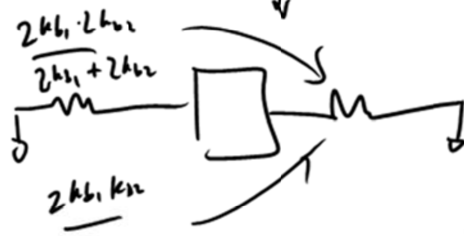




The stiffnesses add for springs in parallel



$$\frac{2k_{b1}\cdot 2k_{b2}}{2k_{b1}+2k_{b2}}$$



$$\frac{2k_{b1}k_{b2}}{k_{b1}+k_{b2}}$$

$$\boxed{k_{eq} = \frac{4k_{b1}k_{b2}}{k_{b1}+k_{b2}}}$$

$$R_{eq} = \left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_1+R_2}$$

springs in series ↕ R's in parallel

F

x_{total}

k_2

x_1

k_1

$$\boxed{X_1 = X_{total}\frac{k_2}{k_1+k_2}}$$