

EE236
10/6/04
①

Propagation in anisotropic crystals Uniaxial Crystals

Friday:

Begin normal mode expansions

As discussed last time, it is easier to calculate the propagation of waves in anisotropic crystals by using stored energy rather than force and displacement.

The stored energy density is

$$W = \frac{1}{2} \vec{E} \cdot \vec{D}$$

If the media is lossless and not Gyrotropic, we can write

$\vec{D} = \tilde{\epsilon} \vec{E}$ where $\tilde{\epsilon}$ is
diagonal

$$\begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}$$

and in that case we can invert
the relationships

$$D_x = \frac{1}{\epsilon_{xx}} E_x \quad D_y = \frac{1}{\epsilon_{yy}} E_y \\ D_z = \frac{1}{\epsilon_{zz}} E_z$$

(2)

and so we can write

$$\omega = \frac{1}{2} \left(\frac{1}{\epsilon_{xx}} D_x^2 + \frac{1}{\epsilon_{yy}} D_y^2 + \frac{1}{\epsilon_{zz}} D_z^2 \right)$$

we are then going to try to find solutions of the form

$$\vec{D} = \vec{D}_0 e^{i(\omega t - \vec{K} \cdot \vec{r})}$$

$$\vec{E} = \vec{E}_0 e^{i(\omega t - \vec{K} \cdot \vec{r})}$$

$$\vec{H} = \vec{H}_0 e^{i(\omega t - \vec{K} \cdot \vec{r})}$$

and we know $\nabla \cdot \vec{D} = 0 \rightarrow \vec{K} \perp \vec{D}_0$
 $\nabla \cdot \vec{H} = \nabla \cdot \vec{B} = 0 \rightarrow \vec{K} \perp \vec{H}_0$

using the Maxwell equations

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} = - \mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$-i \vec{K} \times \vec{H}_0 = i \omega \vec{D}_0$$

$$-i \vec{K} \times \vec{E}_0 = -i \omega \mu_0 \vec{H}_0$$

$$i \vec{K} \times \left(\frac{-i \vec{K} \times \vec{E}_0}{-i \omega \mu_0} \right) = i \omega \vec{D}_0$$

(3)

$$-\frac{1}{\omega^2 \mu_0} (\vec{K} \times \vec{K} \times \vec{E}_0) = \vec{D}_0$$

We can use the vector identity,

$$A \times B \times C = B(A \cdot C) - C(A \cdot B)$$

$$-\frac{1}{\omega^2 \mu_0} (\vec{K}(\vec{K} \cdot \vec{E}_0) - \vec{E}_0 |\vec{K}|^2) = \vec{D}_0$$

If we separate \vec{E}_0 into two parts, that transverse to \vec{K} and that parallel to \vec{K} ,

$$\vec{E}_0 = \vec{E}_{\text{trans}} + E_{\parallel K} \frac{\vec{K}}{|\vec{K}|}$$

The above reduces to

$$+\frac{|\vec{K}|^2}{\omega^2 \mu_0} \vec{E}_{\text{trans}} = \vec{D}_0$$

$$\text{Since } \frac{|\vec{W}|^2}{|\vec{K}|^2} = v_{\text{phase}}^2$$

$$\frac{\omega^2}{|\vec{K}|^2} = \frac{c^2}{n^2}$$

$$\vec{D}_0 = \frac{n^2}{c^2 \mu} \vec{E}_{\text{trans}}$$

where
is the index
of refraction

(4)

Since D is \perp to \vec{K} , if we take the dot product of both sides with respect to \vec{D}_0

$$\vec{D}_0 \cdot \vec{D}_0 = \frac{n^2}{c^2 \mu} (\vec{E}_{\text{true}} \cdot \vec{D}_0)$$

$$|D_0|^2 = \frac{n^2}{c^2 \mu} \vec{E}_0 \cdot \vec{D}_0$$

$$\text{since } c^2 = \frac{1}{\mu \epsilon_0}$$

this can also be written

$$(\vec{D}_0)^2 = n^2 \epsilon_0 \vec{E}_0 \cdot \vec{D}_0$$

by writing

$$\vec{D} = \tilde{\epsilon} \vec{E} = \epsilon_0 \epsilon_r \vec{E}$$

$$D_x = \epsilon_x E_x$$

$$E_x = \frac{1}{\epsilon_x} D_x$$

$$D_y = \epsilon_y E_y$$

$$E_y = \frac{1}{\epsilon_y} D_y$$

$$D_z = \epsilon_z E_z$$

$$E_z = \frac{1}{\epsilon_z} D_z$$

(5)

we find

$$\vec{D}_0 = n^2 \epsilon_0 (\vec{E} - \vec{E} \cdot \frac{\vec{K}}{|\vec{K}|} \frac{(\vec{K})}{|\vec{K}|})$$

$$= n^2 \epsilon_0 \hat{x} (E_x - (\vec{E} \cdot \frac{\vec{K}}{|\vec{K}|}) \frac{K_x}{|\vec{K}|})$$

$$+ n^2 \epsilon_0 \hat{y} (E_y - (\vec{E} \cdot \frac{\vec{K}}{|\vec{K}|}) \frac{K_y}{|\vec{K}|})$$

$$+ n^2 \epsilon_0 \hat{z} (E_z - (\vec{E} \cdot \frac{\vec{K}}{|\vec{K}|}) \frac{K_z}{|\vec{K}|})$$

$$E_x = n^2 \epsilon_0 / \epsilon'_x (E_x - (\vec{E} \cdot \frac{\vec{K}}{|\vec{K}|}) \frac{K_x}{|\vec{K}|})$$

and a similar result for the
 $y + z$ components

solving for E_x

$$E_x = \frac{n^2 \frac{K_x}{K} (\frac{\vec{K}}{K} \cdot \vec{E})}{n^2 - \epsilon'_x}$$

$$\text{where } \epsilon'_x = \frac{\epsilon_x}{\epsilon_0}$$

If we take the vector for \vec{E} defined above and dot it with $\frac{\vec{K}}{|K|}$

$$\frac{\vec{K}}{|K|} \cdot \vec{E} = \frac{K}{|K|} \cdot \vec{E} \sum_{K=x,y,z} \frac{n^2 \left(\frac{K_x}{|K|} \right)^2}{n^2 - \epsilon'_K}$$

or

$$\sum_{K=x,y,z} \frac{n^2 \left(\frac{K_x}{K} \right)^2}{n^2 - \epsilon'_K} = 1$$

which is called the Fresnel equation

which gives us the index of refraction as a function of the direction of \vec{K}

$$\left(\frac{K_x}{|K|}, \frac{K_y}{|K|}, \frac{K_z}{|K|} \right)$$

This method of solving for the propagation in a crystal is simplified by the method of the Index ellipsoid.

7

If you find the index

ellipsoid

$$\frac{D_x^2}{\epsilon_x} + \frac{D_y^2}{\epsilon_y} + \frac{D_z^2}{\epsilon_z} = 2W_0E_0$$

can be used together with the fact that $D \perp R$ to find two polarizations for D which can propagate without rotation

we find the ellipsoid

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

and then cut a plane \perp to R

$$\vec{r} \cdot \vec{s} = x s_x + y s_y + z s_z = 0$$

this forms the equations for an ellipse

If you find the extrema of the ellipse, these directions are the directions of D for the two modes, and the radius is $\frac{1}{n^2}$ for each of those modes.

(5)

this turn out to be easy if two of the direction have the same index

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_c^2}$$

where θ is the angle of R from the optical axis