

EE276
9/29/09

(1)

In the next few lectures, we will be looking at several classical EM problems which have bearing on Optical interactions & devices

- Kramers-Kronig relationship
(relationship between real & imaginary part of χ)

- Propagation in anisotropic crystals
(needed for modulators & nonlinear devices)

- Modes in a cavity
- Black body radiation
- Interactions with photons
- Spontaneous emission

With the last lecture, we ended with the average power transferred to electrons with the polarization

$$\text{Power} = \frac{\epsilon_0}{\omega} \int_{-\infty}^{\infty} E^2 dE$$

$$\text{with the result Power} = \frac{\omega}{2} \epsilon_0 |E|^2 R_{\text{cav}}(ik_c)$$

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$$= \eta_0 \operatorname{Re}(i\omega_0 \chi_e E E^*)$$

$$= \frac{\omega}{2} \epsilon_0 |E|^2 \operatorname{Re}(i\chi_e)$$

So the power transferred is proportional to the imaginary part of χ_e

We now write

$$\chi_e = \chi'_e - i\chi''_e \quad (\chi'_e + \chi''_e \text{ real})$$

+ we get $\frac{\text{Power}}{\text{Volume}} = \frac{\omega \epsilon_0 \chi''_e}{2} |E|^2$

if we are far from resonance of the Q.M. system, the average power will be low and $\chi'' \approx 0$

We then have $\vec{P} = \epsilon_0 \vec{\chi}'_e \vec{E}$ if isotropic
+ not optically active

+ we can write

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E}$$

$$\text{or } \vec{D} = \epsilon \vec{E}$$

\Rightarrow Valid only for a single frequency

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So the \vec{E} field perturbs the QM system, which then has an average expectation value $\langle \vec{r} \rangle$ for the position of the electrons which is nonzero, which changes the propagation of the \vec{E} field.

If we then look back at M.E.s

$$\nabla \times \vec{H} = \vec{i}_{\text{free}} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = \rho_{\text{free}}$$

$$\vec{D} = \epsilon \vec{E}$$

$$+\vec{B} = \mu \vec{H}$$

\rightarrow they are valid only at a single frequency

\rightarrow all of the S problem for bound charge is hidden (where does it go?)

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the equation $P = \int_0^\infty G(z) E(\vec{r}, t-z) dz$
 (for a time invariant, linear + causal relationship)
 is a special case, when transformed
 as discussed earlier, we have

$$E(w) = E_0 + \int_0^\infty G(z) e^{izw} dz$$

and so we have

$$\tilde{P}_w = E_w \tilde{E}$$

→ E_w is now restricted to the
 isotropic, not optically active, case
 so it is a complex scalar

The Kramers-Kronig relations are
 a consequence of these approximations.

Kariv treats this in appendix I,
 or better, see Jackson 3rd ed pp 332

This follows from the realization that
 $E(w)$ is a complex variable
 of the complex variable w , and is
 is an analytic function of w
 in the upper half plane

We also need the requirement $G(z) \rightarrow 0$
 as $z \rightarrow \infty$, so the material
 has no "memory"

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Note that the equation

$$\vec{P} = \int_0^\infty Q(z) \vec{E}(\vec{r}, t-z) dz$$

is not in phasor notation $\rightarrow Q(z)$
must be a real function ($\vec{E} + \vec{P}$ are real)

So when we look at

$$\epsilon = \epsilon_0 + \int_0^\infty G(z) e^{i\omega t} dz$$

$$\epsilon^* = \epsilon_0 + \int_0^\infty G(z) e^{-i\omega t} dz$$

$$\Rightarrow \epsilon(\omega) = \epsilon^*(\omega^*)$$

Note that a pole in the upper half plane corresponds to a complex frequency

$$\omega = \omega_r + i\gamma$$

where there is a potentially $\propto \vec{P}$
in the presence of a decaying \vec{E} field

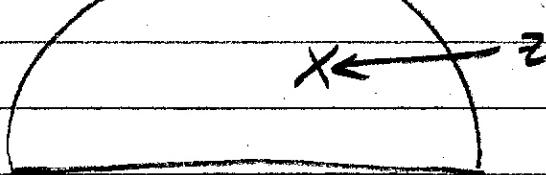
$$\vec{E} = E(r) e^{i(\omega_r + \gamma)t}$$

$$= E(r) e^{i\omega_r - \gamma t}$$

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This is all simply a consequence of the fact that what we see as two very different things—(gain or loss) vs index of refraction, are due to the same physical process

Now consider the integral along a contour around the ~~upper~~ complex plane:



$$\int_{-\infty}^{\infty} \frac{\epsilon(w) - \epsilon_0}{w' - z} dw'$$

because this is an analytic expression except for the pole at $w' = z$, the result of the contour integration is the residue at the pole

$$\int_{-\infty}^{\infty} \frac{\epsilon(w') - \epsilon_0}{w' - z} dw' = 2\pi i [\epsilon(z) - \epsilon_0]$$

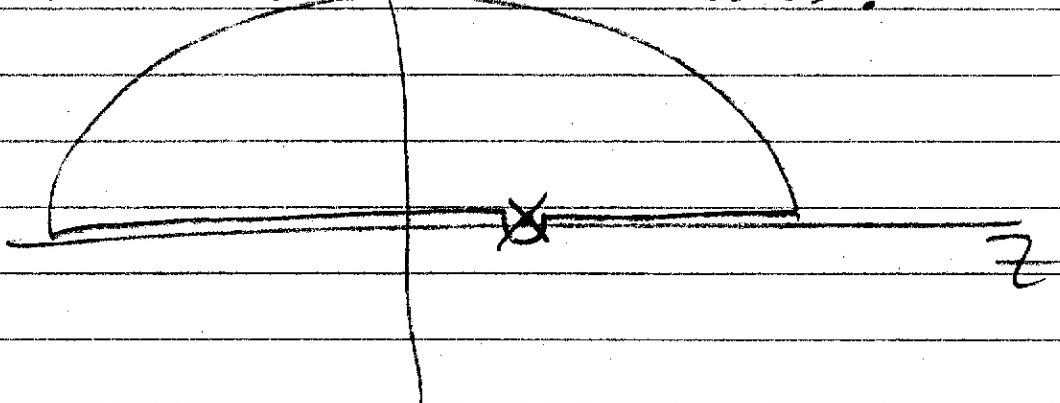
So we get

$$\epsilon(z) = \epsilon_0 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\epsilon(w')/\epsilon_0 - 1]}{w' - z} dw'$$

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now this expression is valid anywhere in the upper half plane, so we take the limit as the pole approaches the axis where z ($\sim w$) is real.

The contour then becomes:



This integral has three parts, an integral along the real axis approaching the pole symmetrically, the infinitesimal integral around the bottom of the pole, and the semicircular integral which approaches infinity.

The semicircular integral at infinity goes to zero on physical grounds (Matter cannot respond infinitely fast)
(the polarization response to Gamma rays is very small)

The integral along the real axis becomes a Principal value integral from $-\infty$ to $+\infty$, and the integral around the infinitesimal contour can be evaluated to $\frac{1}{2}$ the residue of the pole \rightarrow

$$\mathcal{E}(w) = \mathcal{E}_0 + \frac{1}{\pi i} \text{P.V.} \int_{-\infty}^{\infty} \frac{[\mathcal{E}(z) - \mathcal{E}_0]}{z - w} dz$$

where z is now constrained to be on the real axis

Notice that the i in the denominator means that the integral of the real part of \mathcal{E} allows us to find the imaginary part of \mathcal{E} , and the integral of the imaginary part of \mathcal{E} allows us to find the real part of \mathcal{E} .

$$\text{Re}\{\mathcal{E}(w)\} = \mathcal{E}_0 + \frac{2}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{Im}\{\mathcal{E}(z)\}}{z - w} dz$$

$$\text{Im}\{\mathcal{E}(w)\} = \frac{2w}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{Re}\{\mathcal{E}(z) - \mathcal{E}_0\}}{z - w} dz$$

Multiplying the integrands by

$$\frac{(z+w)}{(z-w)}$$

we get

$$R_c\{\epsilon(\omega)\} = 1 + \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{(z+w) \text{Im}(\epsilon(z))}{z^2 - w^2} dz$$

$$\text{Im}\{\epsilon(\omega)\} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(z+w) [R_c\{\epsilon(z)\} - \epsilon_0]}{z^2 - w^2} dz$$

now the integrals are expressed in terms
of functions which are even [$R_c\{\epsilon(z)\}$]
or odd [$\text{Im}(\epsilon(z))$, z] $(z^2 - w^2)$,

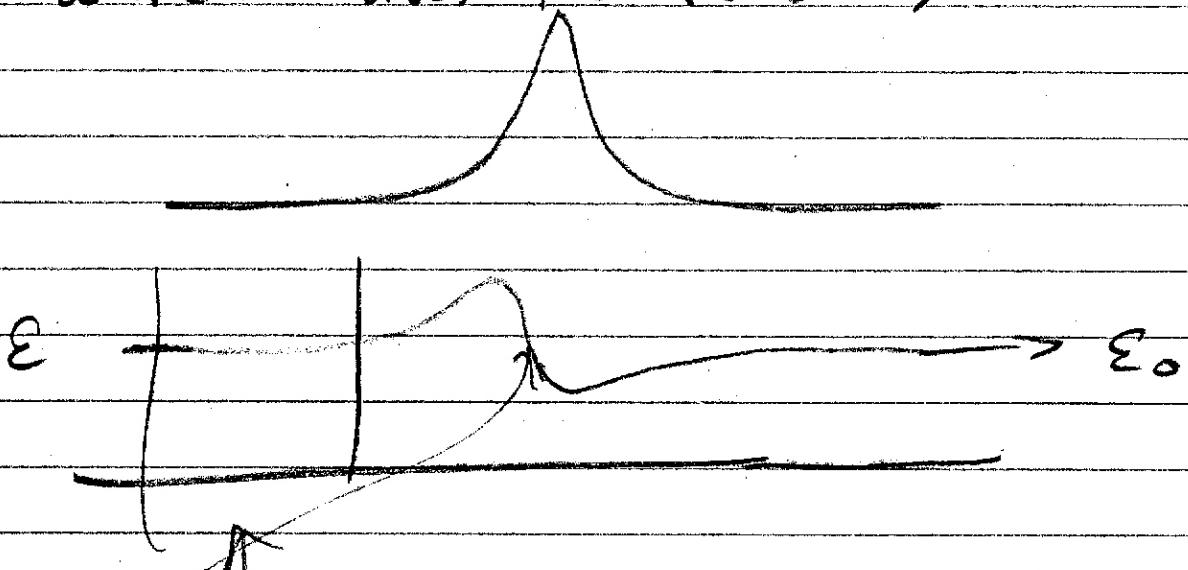
so we can write the integrals

$$R_c\{\epsilon(\omega)\} = 1 + \frac{2}{\pi} \text{P.V.} \int_0^{\infty} \frac{z' \text{Im}(\epsilon(z))}{z'^2 - w^2} dz$$

+

$$\text{Im}\{\epsilon(\omega)\} = -\frac{2w}{\pi} \text{P.V.} \int_0^{\infty} \frac{\text{Re}\{\epsilon(z) - \epsilon_0\}}{z'^2 - w^2} dz$$

thus if we have a material which is strongly absorbent over a narrow range of frequencies, we get an dielectric constant which does the following:



In this range, ϵ is approximately independent of frequency

this region is called anomalous dispersion

It turns out that there are many interesting + useful special cases for χ

If χ is real + diagonalizable, but the diagonal elements are all different, this is called an anisotropic, non optically active lossless material

$$+ \text{ we have } D = \sum \epsilon_k e_k E_k$$

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

for a specific coordinate system called the principle dielectric coordinate system.

If any two of the diagonal elements are now equal, this is called a uniaxial crystal, + whichever direction is different is called the principle optical axis