EE236
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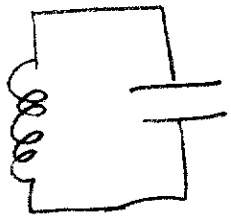
The Harmonic Oscillator



the Classical energy of an L-C oscillator circuit is:

$$W_{total} = W_{cap} + W_{ind}$$

$$= \tfrac{1}{2} C V^2 + \tfrac{1}{2} L I^2$$

$$I = C \frac{\partial V}{\partial t}$$

Classically the voltage & current will oscillate in tandem at a frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

the State of the system is 1-D, just like a particle in 1-D, and the classical equation of motion is the same as that for a particle in a 1-D parabolic potential well.

So we are going to assume that a complete description of the state of the system will be a function

$\psi(V)$ — a probability amplitude over the voltage

to find the operators which correspond to "current" and Energy, we use the similarity of the dynamical variables P_x & x to those for I & V to try and find an operator for I:

$$I \rightarrow \hat{I} = -\left(i\hbar\omega_0 \frac{\partial}{\partial V}\right)$$

where the constants come from dimensional analysis

$$\hat{H} = \frac{1}{2} C V^2 + \frac{1}{2} L\omega_0^2 \left(-i\hbar \frac{\partial}{\partial V}\right)^2$$

$$\hat{H} = \frac{1}{2} C V^2 - \frac{\hbar^2 L \omega_0^2}{2} \frac{\partial^2}{\partial V^2}$$

We then have the equation of motion

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$$

$$i\hbar \frac{\partial}{\partial t} \psi(V,t) = \left(\frac{1}{2}CV^2 - \frac{\hbar^2 L \omega_0^2}{2} \frac{\partial^2}{\partial V^2}\right) \psi(V,t)$$

Notice that $\psi(V)$ can be any function of V, and it will change in time according to this differential equation

We can then use the method of separation of variables to solve this D.E. in V & t. If we look for solutions of the form

$$\psi(V,t) = \phi(V)\, e^{-i\frac{E}{\hbar}t}$$

then we get the Hamiltonian eigenstate equation

$$\hat{H} |\phi\rangle = E |\phi\rangle$$

$$i\hbar \left(-i\frac{E}{\hbar}\right) \phi(V) e^{-i\frac{E}{\hbar}t} =$$

$$\left(\frac{1}{2}CV^2 - \frac{\hbar^2 L \omega_0^2}{2} \frac{\partial^2}{\partial V^2}\right) \phi(V) e^{-i\frac{E}{\hbar}t}$$

$$E\,\phi(v) = \left(\frac{1}{2} C v^2 - \frac{\hbar^2 L \omega_0^2}{2} \frac{\partial^2}{\partial v^2}\right)\phi(v)$$

If we change variables, we can put this into canonical form

$$\frac{d^2 u(\xi)}{d\xi^2} + (\lambda - \xi^2)\, u = 0$$

using Yariv's notation

There are several ways to find the eigenfunctions of this equation, we will look at an analytical method developed by Hermite, and the raising and lowering operator method.

The method developed by Hermite is to start with a particular solution

$$e^{-(1/2)\xi^2} \qquad \text{(a Gaussian)}$$

and then use the method of variation of parameters

for ξ^2 large $\xi^2 \gg \lambda$
the equation is approximated by

$$\frac{d^2 u}{d\xi^2} - \xi^2 u = 0$$

$$u(\xi) = e^{-(1/2)\xi^2}$$

$$\frac{d\,u(\xi)}{d\xi} = e^{-(1/2)\xi^2}(-1/2)2\xi$$

$$\frac{d^2 u(\xi)}{d\xi^2} = e^{-(1/2)\xi^2}\xi^2 + e^{-1/2\xi^2}$$

but ξ^2 is large

so the asymptotic behavior for ξ large is correct.

if we assume a form

$$u(\xi) = H(\xi)\, e^{-(1/2)\xi^2}$$

and substitute into the above, we get

$$\frac{d^2 H}{d\xi^2} - 2\xi\frac{dH}{d\xi} + (\lambda - 1)H = 0$$

To preserve the correct asymptotic behavior, H will need to grow no faster than some power of ξ, so we try a power series

$$H(\xi) = \xi^s (a_0 + a_1 \xi + a_2 \xi^2 + \dots)$$

we will have to require that the coefficients a_n converge sufficiently fast.

It turns out to do that, the series will have to terminate.

plugging in the expansion, we can set equal powers of ξ equal on both sides of the equation, (because x^n is an orthogonal set)

we then get the equations for a_n:

$$s(s-1)\,a_0 = 0$$

$$(s+1)\,s\,a_1 = 0$$

$$(s+2)(s+1)\,a_2 - (2s+1-\lambda)\,a_0 = 0$$

and so on

We now assume that $a_0 \neq 0$, $a_s = 0$ for some s. The requirement that the series terminates makes this a set of finite polynomials $\rightarrow$ the Hermite polynomials

$$H_0(\xi) = 1$$

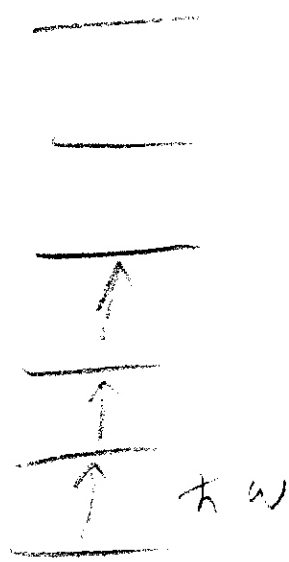
$$H_1(\xi) = 2\xi$$

$$H_2(\xi) = 4\xi^2 - 2$$

and so on

the other method of finding the states of the Harmonic oscillator is the method of raising & lowering operators:

the method of raising & lowering operators uses the fact that there are an ∞ number of closely related states



why should this be so?