

EE 236
9/8/04
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Reading: Chapter 2

Time rate of change of an expectation value

Commutation and uncertainty

Harmonic oscillator

$$\frac{d}{dt} \langle A \rangle = \frac{d}{dt} \langle \Psi | \hat{A} | \Psi \rangle$$

$$= \left(\frac{d}{dt} \langle \Psi | \right) \hat{A} | \Psi \rangle + \langle \Psi | \frac{d}{dt} (\hat{A}) | \Psi \rangle$$

$$= \left(\frac{d}{dt} \langle \Psi | \right) \hat{A} | \Psi \rangle + \langle \Psi | \left(\frac{\partial \hat{A}}{\partial t} \right) | \Psi \rangle$$

$$+ \langle \Psi | \hat{A} \frac{\partial^2}{\partial t^2} | \Psi \rangle$$

if we have a Hamiltonian for this system

$$\hat{H} | \Psi \rangle \equiv i \hbar \frac{\partial}{\partial t} | \Psi \rangle$$

$$\frac{\partial}{\partial t} | \Psi \rangle = - \frac{i}{\hbar} \hat{H} | \Psi \rangle$$

$$\frac{\partial}{\partial t} (\langle \Psi |) = + \frac{i}{\hbar} \langle \Psi | \hat{H}$$

(\hat{H} is Hamiltonian)

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$$\begin{aligned}\frac{d\langle A \rangle}{dt} &= +\frac{i}{\hbar} \langle \psi | \hat{A} \hat{A}^\dagger | \psi \rangle \\ &\quad + \langle \psi | \frac{\partial \hat{A}}{\partial t} | \psi \rangle \\ &= \frac{i}{\hbar} \langle \psi | \hat{A} H | \psi \rangle\end{aligned}$$

if we define the commutator of two operators:

$$[\hat{C}, \hat{D}] = \hat{C}\hat{D} - \hat{D}\hat{C}$$

we have the change in the expectation value of the observable

$$\begin{aligned}\frac{d}{dt} \langle \psi | \hat{A} | \psi \rangle &= \langle \psi | \left(\frac{\partial \hat{A}}{\partial t} \right) | \psi \rangle \\ &\quad + \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{A}] | \psi \rangle\end{aligned}$$

for any operator \hat{A} and the Hamiltonian \hat{H}

Note that for any operator without explicit time variation, if it commutes with \hat{A} , its expectation value does not change with time.

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If we look at position and momentum of a free particle, for example

$$\frac{d}{dt} \langle \psi | \hat{x} | \psi \rangle = \frac{i}{\hbar} \langle \psi | [\hat{H}, \hat{x}] | \psi \rangle$$

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$\frac{d}{dt} \langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left[-\frac{i\hbar}{2m} \frac{\partial^2}{\partial x^2} x + \frac{\hbar^2}{2m} x \frac{\partial^2}{\partial x^2} \right] \psi(x) dx$$

$$\frac{d}{dt} \langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left[-\frac{i\hbar}{2m} \frac{\partial}{\partial x} \left(1 + x \frac{\partial}{\partial x} \right) + \frac{\hbar^2}{2m} x \frac{\partial^2}{\partial x^2} \right] \psi(x) dx$$

$$\frac{d}{dt} \langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left[-\frac{i\hbar}{2m} \left(\frac{\partial}{\partial x} + x \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x} \right) + \frac{\hbar^2}{2m} x \frac{\partial^2}{\partial x^2} \right] \psi(x) dx$$

$$= \int_{-\infty}^{\infty} \psi^*(x) \left[-\frac{i\hbar}{m} \frac{\partial}{\partial x} \right] \psi(x) dx$$

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and if we look at the classical value $\langle p \rangle = m \langle \frac{dx}{dt} \rangle$

we see that the operator

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$$

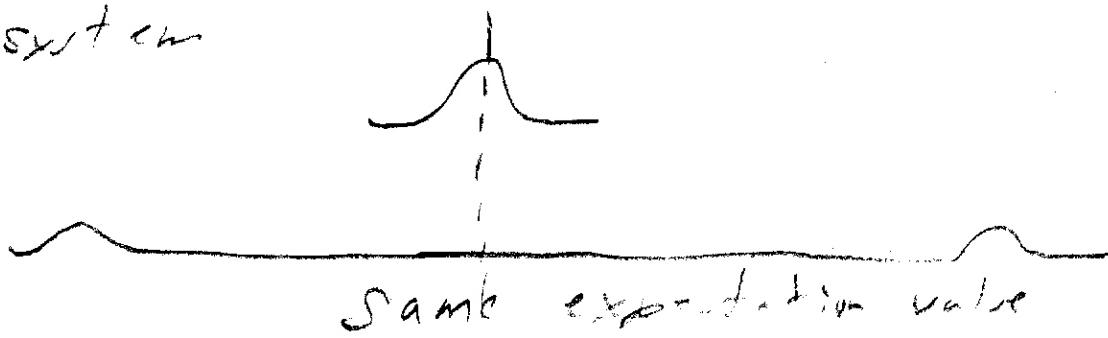
has this expectation value and we will define this as the QM momentum in the x direction, in the $F(x)$ representation

You can also use the $F(p)$ representation, in which the operators for position, momentum become

$$(x) \rightarrow i\hbar \frac{\partial}{\partial p_x}$$

$$(p) \rightarrow p$$

Of course the expectation values don't tell you a lot about the system



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However, if you look at each part of the wave it has the same properties



E.V.(momentum + position
hold for this portion
of the wavefunction as well)

Another observation which can be made is that of mean-square deviation of observations. This just corresponds to another operator

$$\langle \Delta x \rangle^2 = \langle (\hat{x} - \langle x \rangle)^2 \rangle$$

If we look at the momentum and position representations over a 1 degree of freedom system:

$$F(p) = \langle p' | \psi \rangle = h^{-1/2} \int_{-\infty}^{\infty} e^{-ixp/h} f(x) dx$$

$$f(x) = \langle x' | \psi \rangle = h^{-1/2} \int_{-\infty}^{\infty} e^{ixp/h} F(p) dp$$

That is, the functions representing

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the system in the Schrödinger (position) representation, and the function $F(p)$ representing the system in the momentum representation (since the eigenfunctions of \hat{X} in the X representation are the $\delta(X-x_0)$ functions, and the eigenfunctions of \hat{P}_x in the x representation are $e^{ik_x x}$ where $P_x \rightarrow mK_x$)

This can be expressed as one of many uncertainty relationships such as

$$\Delta p_x \Delta x \geq \hbar/2$$

and similar relations.



\rightarrow components of p_x go out to ∞



\rightarrow uniform distribution in p_x , phase carries information about position.

$$[\hat{B}, \hat{C}] \neq 0$$

If any two operators have a non-zero commutator, then

- they do not share a set of eigenstates
- they can not be simultaneously diagonalized
- a simultaneous measurement of both can not be made
- measurements taken in rapid succession of one observable will perturb the eigenstates of the other.

Examples

$$[x, p_x] \neq 0$$

$$\text{but not } [x, y] \text{ or } [x, p_y]$$

$$\text{In general } [x, H] \neq 0$$

$$\text{for a free particle } [p_x, H] = 0$$

The Harmonic oscillator:

The harmonic oscillator is a good model for the Electromagnetic fields of a single mode of a resonator. We will look at it for a L-C oscillator circuit.



The classical energy of an L-C circuit is

$$W_{\text{total}} = W_{\text{cap}} + W_{\text{ind}}$$

$$= \frac{1}{2} C V^2 + \frac{1}{2} L I^2$$

$$\text{where } I = C \frac{dV}{dt}$$

Classically, the voltage will oscillate at a frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

What is the "state" of the L-C circuit at a point in time?

for the L-C circuit, the state of the system becomes a distribution over the voltage of the oscillator

$$\psi(v)$$

now we need to figure out what operator corresponds to the observable current

In analogy to momentum, we will try the operator

$$\hat{I} = -\left(i\hbar \frac{\partial}{\partial v}\right) \omega_0$$

from which we get the Hamiltonian

$$H = \frac{1}{2} C V^2 + \frac{1}{2} L \omega_0^2 \left(-i\hbar \frac{\partial}{\partial v}\right)^2$$

now let's use the classical oscillator frequency to express this equation:

$$H = \frac{1}{2} C V^2 + \frac{1}{2} L \left(-i\hbar \frac{\partial}{\partial v}\right)^2$$

$$H = -\frac{\hbar^2 L \omega_0^2}{2} \frac{\partial^2}{\partial v^2} + \frac{1}{2} C V^2$$