Repeated Games with Incomplete Information

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Bayesian Collective Choice Problem

\((C, \Theta_1, \ldots, \Theta_n, U_1, \ldots, U_n, P)\)

- \(C\) - set of available strategies
- \(\theta_i \in \Theta_i\) - type of player \(i\)
- \(U_i(c, \theta_1, \ldots, \theta_n)\) - payoff of the player \(i\) given the strategy \(c \in C\)
- \(P(\theta_1, \ldots, \theta_n)\) - probability that \((\theta_1, \ldots, \theta_n)\) is true
• $\pi(c|s_1, ..., s_n) \geq 0$ - choice mechanism
  \[ \sum_{c \in C} \pi(c|s_1, ..., s_n) = 1 \]
  for every $(s_1, ..., s_n) \in S_1 \times ... \times S_n$

• $S_i$ - set of possible responses of the player $i$

• $\tau_i : \Theta_i \mapsto S_i$ - truthful response map

• If $S_i = \Theta_i$ then $S_i$ - standard response sets

Allocation of conditionally-expected payoffs:

$$V(\pi) = ((V_1(\pi|\theta_1))_{\theta_1 \in \Theta_1}, ..., (V_n(\pi|\theta_n))_{\theta_n \in \Theta_n})$$

where $V_i(\pi|\theta_i) = Z_i(\pi, \theta_i|\theta_i)$

Feasible set of expected allocations:

$$F = \{V(\pi) : \pi \text{ is a choice mechanism}\}$$

Incentive feasible set of expected allocations:

$$F^* = \{V(\pi) : \pi \text{ is Bayesian incentive compatible}\}$$

**Theorem:**

$F^* \subset F$ is nonempty, convex and compact
Can we achieve this in *Repeated Game*?

Friedman’s Theorem

Let $G$ be a single-stage game and $(e_1, ..., e_n)$ denote the payoff from a Nash equilibrium of $G$.

If $\bar{x}=(x_1, ..., x_n)$ is a feasible payoff from $G$ such that $x_i \geq e_i \forall i$, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game of $G$ which achieves $\bar{x}$, provided that discount factor $\delta$ is close enough to one.

James W. Friedman

“A Non-cooperative Equilibrium for Supergames”,
*Review of Economic Studies* 38/1 (1971)
Repeated games: what can happen?

1. Arbitrator and players do not learn
2. Arbitrator does not learn but the players do
3. Arbitrator does learn but the players do not
4. Arbitrator and players do learn

Arbitrator and players do not learn

Makes sense to treat the repeated game as series of independent one-shot games, in which we look for equilibrium
Response Plan Equilibrium

- $\sigma_i$ - response plan of player $i$ - maps $\theta_i \in \Theta_i$ onto probability distribution over $S_i$
  \[ \sum_{s_i \in S_i} \sigma_i(s_i|\theta_i) = 1, \quad \sigma_i(s_i|\theta_i) \geq 0 \]
- Expected payoff for player $i$:
  \[ W_i(\pi, \sigma_1, \ldots, \sigma_n|\theta_i) = \sum_{\alpha \in \Theta_1 \times \ldots \times \Theta_n} \sum_{s_s \in S_1 \times \ldots \times S_n} \sum_{c \in C} p_i(\alpha|\theta_i) \left( \prod_{j=1}^{n} \sigma_j(s_j|\theta_j) \right) \pi(c|s) U_i(c|\alpha) \]
  \[(\sigma_1, \ldots, \sigma_n): W_i(\pi, \sigma_1, \ldots, \sigma_i, \ldots, \sigma_n|\theta_i) \geq W_i(\pi, \sigma_1, \ldots, \sigma',\ldots, \sigma_n|\theta_i)\]

Allocation of conditionally-expected payoffs:
\[ W(\pi, \sigma_1, \ldots, \sigma_n) = \left( \left( W_i(\pi, \sigma_1, \ldots, \sigma_n|\theta_i) \right)_{\theta_i \in \Theta_i} \right)_{i=1}^{n} \]

Equilibrium feasible set of expected allocations:
\[ F^{**} = \{ W(\pi, \sigma_1, \ldots, \sigma_n) : (\sigma_1, \ldots, \sigma_n) \text{ is response plan equilibrium for } \pi \} \]

Theorem: $F^{**} = F^*$
Arbitrator does not learn but the players do

The answer to this problem is given by

Ehud Kalai, Ehud Lehrer
“Rational Learning Leads to Nash Equilibrium”
Econometrica 61/5 (1993)

Arbitrator does learn

**Patience** – what is it?

- Patient Arbitrator: if outcome functions \( c(\theta) = c'(\theta) \) after certain \( T \), then they are treated as equal
- Patient Player: discounting factor \( \delta \to 1 \)
Arbitrator does learn

**Idea 1:** the arbitrator has to force the players to reveal their true types. The game starts with the sequence of incentive compatible rounds. The length of the sequence depends on $\max\{\delta_i\}_n$. In the infinite game the arbitrator can make this sequence as long as needed. After that the arbitrator makes Pareto optimal choice using the types that were revealed by the players.

Arbitrator does learn

**Idea 2:** let each player be a dictator for $d_i$ number of rounds

- $\theta$ and $\theta'$ are distinguishable types for the player $i$ if $c(\theta) > c(\theta')$ and $c'(\theta) < c'(\theta')$
- $d_i =$ number of pair-wise distinguishable types for the $i$-th player
- Arbitrator has all these pairs enumerated and at every dictatorship round of the player $i$ this player is offered only two choice alternatives – each giving him better payoff if he were of the corresponding type
- After all the dictatorships there is a period of conflict outcomes, which duration depends on $\max\{\delta_i\}_n$
- After that the arbitrator makes Pareto optimal choice using the types that were revealed by the players during their dictatorships
Open Question

Which Pareto optimal allocation should the arbitrator choose?

What if everybody learns, but the players are patient and the arbitrator is not?

The repeated game turns into the sequence of one-shot games
References


2. D. Fudenberg, E. Maskin “The Folk Theorem in Repeated Games with Discounting or with Incomplete Information”, *Econometrica* 54/3 (1986)


