EE228a Communication Networks

Alessandro Abate
aabate@eecs.berkeley.edu

A Duality Model of TCP and Queue Management Algorithms
Steven H. Low
Goals

- Introduce a duality model for end-to-end congestion control;
- Reinterpret the equilibrium properties of the network through primal and dual variables;
- Compare different existing protocols under this new framework.
Tools

- **Communication Networks**: concepts, functions, goals;
- **Convex Optimization**: duality theory;
- **Internetworking**: TCP (sources) and AQM (links) schemes.
Network Model

$L$ links; $S$ sources. Each $s$ uses a set $L_s$.

$$R_{ls} = \begin{cases} 1, & \text{if } l \in L_s \\ 0, & \text{else} \end{cases}, \quad \text{routing matrix}$$

Source $s \rightarrow x_s(t)[\text{packets/sec}]$, transmission rate.

Link $l \rightarrow p_l(t)$, congestion measure; $c_l$, capacity.

Aggregate source rate at link $l$: $y_l(t) = \sum_s R_{ls} x_s(t)$.

End-to-end congestion measure for $s$: $q_s(t) = \sum_l R_{ls} p_l(t)$. 
Network’s dynamics

\[ x_s(t + 1) = F_s(x_s(t), q_s(t)); \]

\[ p_l(t + 1) = G_l(y_l(t), p_l(t), v_l(t)); \]

\[ v_l(t + 1) = H_l(y_l(t), p_l(t), v_l(t)). \]

Assumption: \( \exists \) an equilibrium set for \((x, p)\)

{we'll consider the positive quadrant}.

Assumption2: \( F_s \in C^1; \partial F_s / \partial q_s \neq 0 \) in an open set \( A \).

Then, by the implicit function theorem, \( \exists f_s \in C^1, s.t. \)

\[ q_s = f_s(x_s) > 0, \text{ around } an \text{ equilibrium point.} \]
Implicit Function Theorem

Let $A$ be an open set in $\mathbb{R}^{n+k}$ and let $f : A \rightarrow \mathbb{R}^n$ be a $C^r$ function. Write $f$ in the form $f(x,y)$, where $x$ and $y$ are elements of $\mathbb{R}^k$ and $\mathbb{R}^n$. Suppose that $(a, b)$ is a point in $A$ such that $f(a, b) = 0$ and the determinant of the $n \times n$ matrix whose elements are the derivatives of the $n$ component functions of $f$ with respect to the $n$ variables, written as $y$, evaluated at $(a, b)$, is not equal to zero. The latter may be rewritten as $\text{rank}(Df(a, b)) = n$. 
Implicit Function Theorem, cont’d

Then there exists a **neighborhood** $B$ of $a$ in $\mathbb{R}^m$ and a unique $C^r$ function $g: B \to \mathbb{R}^e$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in B$. 
Utility Functions and Optimization Problem

Each source $s$ has a utility function:

$$U_s(x_s) = \int f_s(x_s) \, dx_s, \quad x_s \geq 0.$$  

By assumption, $U_s$ is non-decreasing (higher rate, higher utility), other than continuous. Moreover, assuming $f_s$ non-increasing (higher congestion, lower bit rate), then $U_s$ is concave.

Problem: Maximize Aggregate Utility,

$$\max \sum_{x \geq 0} U_s(x_s), \quad \text{s.t. } Rx \leq c.$$
Optimization Problem

Constraints: flow rate $y_i$ doesn’t exceed capacity $c_i$.

The solution exists (continuous objective and compact feasibility set) and is unique if the objective is strictly concave; but difficult to compute: the sources are coupled through the shared links.
Duality Theory

Given an optimization problem,

\[ p^* = \min f_0(x), \]

s.t. \( f_i(x) \leq 0, \ i = 1, \ldots, m \)

Define the Lagrangian function,

\[ L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x), \quad \lambda_i > 0. \]

Minimize the Lagrangian,

\[ g(\lambda) = \inf_{x} L(x, \lambda) = \inf_{x} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right) \]

and define the dual function,

\[ d^* = \max_{\lambda} g(\lambda). \quad \text{Usually } d^* < p^* \text{ (weak duality).} \]
Duality Model of TCP/AQM

Let’s build up the Lagrangian,

$$L(x, \lambda) = \sum_s U_s(x_s) - \lambda (Rx - c) = \sum_s \left( \int f_s(x_s) dx_s - \sum_l \lambda_l R_{ls} x_s \right) + \sum_l \lambda_l c_l$$

Interpret the dual variables as congestions in the network:

$$\lambda_l = p_l. \text{ Then } \sum_l R_{ls} \lambda_l = q_s.$$ 

We obtain:

$$d^* = \min_p \sum_s \max_{x_s \geq 0} (U_s(x_s) - x_s q_s) + \sum_l p_l c_l$$

The equilibrium $$(x, p)$$ is actually a solution of the primal/dual problem.
Existence and Uniqueness of a solution

C1: $F_s$ and $G_i$ are non-negative;
C2: implicit function theorem is valid;
C3: for every link, $y_i < c_i$, with equality if $p_i > 0$;
C4: $f_s$ are strictly decreasing.

**Thm**: Assume C1, C2 hold. The problem is dual-feasible iff C3 holds. If C4 holds, then the optimum is unique.
TCP/AQM protocols can be modeled as primal/dual algorithms. \( U_s \) depends on TCP, \( F_s \). AQM \((G,H)\) ensures (strong) duality (match btw input rate and capacity).

Some concepts:
1. **Window**, \( w_s(t) \) (available space at a source);
2. **Round trip time**, \( D_s \) (source-to source travel time);
3. **Mark** (tag attached to information packets which detects a congestion).
4. **Throughput** \( s(t) = \frac{w_s(t)}{D_s} \).
Reno-1

Window size is halved every time a mark is detected, otherwise increased of one unit (additive increase, multiplicative decrease).

$p_l(t)$: marking probability at link $l$, at time $t$.

$$q_s(t) = \sum_l R_{ls} p_l(t) = 1 - \prod_{l \in L_s} (1 - p_l(t)) \approx \sum_{l \in L_s} p_l(t), \text{end-to-end.}$$

Net change in window size:

$$x_s(t)(1 - q_s(t)) \frac{1}{w_s(t)} - x_s(t)q_s(t) \frac{1}{2} \frac{4w_s(t)}{3}$$

Defining $x_s(t) = \frac{w_s(t)}{D_s}$, we find $F_s(x_s(t), q_s(t))$. 
Window size is halved if a mark is detected in a round trip time, otherwise increased.

Define \( q_s(t) = w_s(t)q_s(t) \);

average change in window size:

\[
\frac{1}{D_s}(1 - \hat{q}_s(t)) - \frac{2w_s(t)}{3D_s} \hat{q}_s(t)
\]

From here, define \( F_s(.) \).

Bottom line: at this point, it’s possible to define the utility functions \( U_s(.) \).
RED

Random Early Detection: two internal variables, queue length $b_i(t)$ and average queue length $r_i(t)$. $p_i(t)$ is a function of $b_i(t)$ and $r_i(t)$.

Defining equations for their update, plus a marking probability, we get to the model for $(G,H)$.
REM

Random Exponential Marking: two internal variables, $b_i(t)$, and "price" $r_i(t)$. 
Duality applied to TCP/AQM schemes

It’s possible to derive a function $F$ for Reno-1, Reno-2, as well as functions $G, H$ for RED, REM.

They all satisfy $C1, C2, C4$ of the Theorem. Also $C3$ is valid.

Then, we can solve the optimization problem exploiting fast primal/dual techniques, being sure to eventually attain the optimum.
Study of Interaction

Duality models provide a way to study the interaction between different TCP/AQM schemes.

Starting from a triad \((F,G,H)\), we compute loss, delay, queue length of a network.

Varying specific network’s parameters, we can benchmark performances of different algorithms (only requirement: same congestion measure).
Next Step

After studying the equilibrium properties of different TCP/AQM schemes, also the stability and the dynamics of these protocols have to be assessed.