Combinatorial Auctions: A Survey

Sven de Vries
Rakesh Vohra

Rahul Tandra — EE 228 presentation
Outline of the Presentation

- Define combinatorial auctions
- Formulate it as an integer program
- Set Packing Problem (SPP)
- Complexity issues of SPP
- Solvable instances of the SPP
Combinatorial Auctions

- What are combinatorial auctions?
- Why are they important?
  1) Complementarities
  2) Substitution effects
- Examples of Combinatorial Auctions
  1) FCC spectrum auction
  2) Airport time slots
  3) Railroad segments and delivery routes
Combinatorial Auction Problem (CAP)

- **Problems faced:**
  1) Bidder must submit a bid for every subset of object he is interested in
  2) The bidding function should be transmitted to the auctioneer.

- **Best way out:**
  1) Restrict the kinds of combinations that bidders may bid on

- **Goal:**
  1) To find the winning set of bids.
Problem Formulation: CAP1

- N - set of bidders
- M - set of ‘m’ distinct objects
- Let $b^j(S)$ be the bid announced by agent ‘j’ for the subset ‘S’.
- Let $b(S) = \max_{j \in N} b^j(S)$
- The CAP can be formulated as:

$$\max \sum_{S \subset M} b(S) x_S$$

subject to

$$s.t. \sum_{S \ni i} x_S \leq 1 \forall i \in M$$

$$x_S = 0, 1 \forall S \subset M$$
Issues with CAP1

- CAP1 is a correct model when the bid functions are superadditive, i.e.,
  \[ b^j(A) + b^j(B) \leq b^j(A \cup B) \quad \forall j \in N \quad \text{and} \]
  \[ A, B \subset M \quad \text{such that} \quad A \cap B = \emptyset \]

- This corresponds to the case when the goods complement each other.

- CAP1 fails when the good are substitutes,
  \[ b^j(A) + b^j(B) \geq b^j(A \cup B) \]
Let $y(S, j) = 1$ if the bundle $S \subseteq M$ is allocated to $j \in N$ and zero otherwise.

$$\max \sum_{j \in N} \sum_{S \subseteq M} b^j(S) y(S, j)$$

$$s.t. \sum_{S \ni i} \sum_{j \in N} y(S, j) \leq 1 \forall i \in M$$

$$\sum_{S \subseteq M} y(S, j) \leq 1 \forall j \in N$$

$$y(S, j) = 0, 1 \forall S \subseteq M, j \in N$$
The Set Packing Problem (SPP)

- It is a well studied integer program
- ‘M’ be a set of elements
- ‘V’ be a collection of subsets of ‘M’ with non-negative weights
- To find:
  Largest weight collection of subsets that are pairwise disjoint.
SPP: formulation

Let \( x_j = 1 \) if the \( j^{th} \) set in ‘\( V \)’ with weight \( c_j \) is selected and \( x_j = 0 \) otherwise.

Define \( a_{ij} \) to be 1 if the \( j^{th} \) set in ‘\( V \)’ contains element \( i \in M \)

\[
\begin{align*}
\text{max} & \quad \sum_{j \in V} c_j x_j \\
\text{s.t.} & \quad \sum_{j \in V} a_{ij} x_j \leq 1 \quad \forall \ i \in M \\
& \quad x_j = 0, 1 \quad \forall \ j \in V
\end{align*}
\]
Complexity of the SPP

How hard is the SPP to solve?

1) No polynomial time algorithm for the SPP is known.

SPP belongs to an equivalence class of problems call NP-hard.

Thus, effective solution procedures of CAP rely on 2 things

1) The number of distinct bids should not be large
2) The underlying SPP should be solved reasonably quickly
Solvable Instances of the SPP

- Consider the case when the polyhedron \( P(A) \) has integral extreme points.
- In this case the SPP turns out to be a linear program.
- Thus, can be solved in polynomial time.
- Polyhedron with all integral extreme points is called ‘INTEGRAL’.
Sufficient conditions

- These involve restrictions on the constraint matrix ‘A’. (nothing but restrictions on the number of subsets)
- The most common condition is ‘Total Unimodularity’ (TU).
- Defn: A matrix is TU if the determinant of every square sub matrix is 0, 1 or -1.
- ‘A’ is TU => \( P(A) \) is integral.
Characterization of TU

**Theorem:** Let $B$ be a matrix each of whose entries is 0,1 or -1. Suppose each subset $S$ of columns of $B$ can be divided into two sets $L$ and $R$ such that

$$\sum_{j \in S \cap L} b_{ij} - \sum_{j \in S \cap R} b_{ij} = 0,1 \forall i$$

then $B$ is TU. The converse is also true.
Network Matrix

- A matrix is called a ‘network matrix’ if each column has at most 2 non-zero entries of opposite sign and absolute value 1.
- A matrix is TU if a restricted set of row and column operations can convert it into a network matrix.
- These operations are negating a row (column) or adding a row (column) to another row.
- TU is an invariant property under these operations.
A 0-1 matrix has a ‘consecutive ones property’ if the non-zero entries in each column occur consecutively.

**Theorem:** All 0-1 matrices with ‘consecutive ones property’ are TU.

**Example:** Suppose that the objects auctioned are parcels of land along the shore line (linear order among the parcels). In this case the obvious set of combinations would be contiguous.

Thus the constraint matrix ‘A’ would have the *consecutive ones property*. 
Balanced Matrices

A 0-1 matrix is ‘Balanced’ if it has no square submatrix of odd order with exactly two 1’s in each row and column.

**Theorem**: Let B be a balanced 0-1 matrix. Then the following linear program:

\[
\max \left\{ \sum_j c_j x_j : \sum_j b_{ij} x_j \leq 1 \forall i, x_i \geq 0 \forall j \right\}
\]

has an integral optimal solution when ever the \(c_j\)'s are integral.
Instance of balancedness

- Consider a tree ‘T’ with distance function ‘d’.
- For each vertex ‘v’ in T let N(v,r) denote the set of vertices in T that are within a distance ‘r’ of v.
- Assume that the vertices are parcels of land connected by a road network with no cycles.
- Bidders should bid for subsets of parcels of land constrained by N(v,r) for some vertex v and some number r.
- In this case the constraint matrix will be balanced.
- Thus, the CAP can be solved in polynomial time.
Graph Theoretic Methods

- In some situations $P(A)$ is not integral but still the SPP can be solved in polynomial time.
- Here ‘A’ admits a *graph theoretic interpretation*.
- **Most common case**: When each column of $A$ contains atmost two 1’s.
- Can be solved as a *maximum weight matching problem* in a graph.
Remaining part

- Decentralized methods–relaxation techniques.
- Incentive issues.
- Computational experiments and test problems.