A Short Tutorial on Game Theory

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Dept. of EECS, U.C. Berkeley

Outline

Introduction
- What is game theory about?
- Relevance to networking research
- Elements of a game

Non-Cooperative Games
- Static Complete-Information Games
- Repeated Complete-Information Games
- Stackelberg Games

Cooperative Games
- Nash's Bargaining Solution
- Coalition: the Shapley Value

What Is Game Theory About?

To understand how decision-makers interact
A brief history
- 1920s: study on strict competitions
- 1944: Von Neumann and Morgenstern's book
  Theory of Games and Economic Behavior
- After 1950s: widely used in economics, politics, biology...
  - Competition between firms
  - Auction design
  - Role of punishment in law enforcement
  - International policies
  - Evolution of species

Relevance to Networking Research

Economic issues becomes increasingly important
- Interactions between human users
  - Congestion control
  - Resource allocation
- Independent service providers
- Bandwidth trading
- Peering agreements

Tool for system design
- Distributed algorithms
- Multi-objective optimization
- Incentive compatible protocols

Elements of a Game: Strategies

Decision-maker's choice(s) in any given situation
- Fully known to the decision-maker
- Examples
  - Price set by a firm
  - Bids in an auction
  - Routing decision by a routing algorithm
- Strategy space: set of all possible actions
  - Finite vs infinite strategy space
- Pure vs mixed strategies
  - Pure: deterministic actions
  - Mixed: randomized actions
Elements of a Game: Preference and Payoff

- Preference
  - Transitive ordering among strategies
    \( a \gg b, b \gg c \) \( \Rightarrow a \gg c \)
- Payoff
  - An order-preserving mapping from preference to \( \mathbb{R}^+ \)
  - Example: in flow control, \( U(x) = \log(1+x) - px \)

Rational Choice

- Two axiomatic assumptions on games
  1. In any given situation a decision-maker always chooses the action which is the best according to his/her preferences (a.k.a. rational play).
  2. Rational play is common knowledge among all players in the game.

Question: Are these assumptions reasonable?

Example: Prisoners’ Dilemma

<table>
<thead>
<tr>
<th>Prisoner A</th>
<th>Mum</th>
<th>Fink</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mum</td>
<td>-1, -1</td>
<td>-9, 0</td>
</tr>
<tr>
<td>Fink</td>
<td>0, -9</td>
<td>-6, -6</td>
</tr>
</tbody>
</table>

Different Types of Games

- Static vs multi-stage
  - Static: game is played only once
    - Prisoners' dilemma
  - Multi-stage: game is played in multiple rounds
    - Multi-round auctions, chess games
- Complete vs incomplete information
  - Complete info.: players know each others' payoffs
    - Prisoners' dilemma
  - Incomplete info.: other players' payoffs are not known
    - Sealed auctions

Representations of a Game

- Normal- vs extensive-form representation
  - Normal-form
    - like the one used in previous example
  - Extensive-form

Outline

- Introduction
- Complete-Information Strategic Games
  - Static Games
  - Repeated Games
  - Stackeberg Games
- Cooperative Games
  - Nash's Bargaining Problem
  - Coalitions: the Shapley Value
Static Games

- Model
  - Players know each others’ payoffs
  - But do not know which strategies they would choose
  - Players simultaneously choose their strategies
  ⇒ Game is over and players receive payoffs based on the combination of strategies just chosen
- Question of Interest:
  - What outcome would be produced by such a game?

Example: Cournot’s Model of Duopoly

- Model (from Gibbons)
  - Two firms producing the same kind of product in quantities of \( q_1 \) and \( q_2 \), respectively
  - Market clearing price \( p = A - q_1 - q_2 \)
  - Cost of production is \( C \) for both firms
  - Profit for firm \( i \)
    \[
    J_i = p_i q_i - C \quad q_i = (A - q_1 - q_2) q_i - C q_i
    \]
  define \( B = A - C \)
  - Objective: choose \( q_i \) to maximize profit
    \[
    q_i^* = \arg\max_{q_i} (B - q_i - q_j) q_i
    \]

A Simple Example: Solution

- Firm \( i \)’s best choice, given its competitor’s \( q \)
  \[
  q_1^* = (B - q_2)/2 \quad q_2^* = (B - q_1)/2
  \]

Solution to Static Games

- Nash Equilibrium (J. F. Nash, 1950)
  - Mathematically, a strategy profile \((s_1^*, \ldots, s_i^*, \ldots, s_n^*)\) is a Nash Equilibrium if for each player \( i \)
    \[
    U_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \geq U_i(s_1^*, \ldots, s_{i-1}^*, s, s_{i+1}^*, \ldots, s_n^*)
    \]
  for each feasible strategy \( s_i \)
  - Plain English: a situation in which no player has incentive to deviate
  - It’s fixed-point solution to the following system of equations
    \[
    s_i = \arg\max_{s_i} U(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n), \quad \forall i
    \]
- Other solution concepts (see references)

An Example on Mixed Strategies

- Pure-Strategy Nash Equilibrium may not exist

<table>
<thead>
<tr>
<th>Head (H)</th>
<th>Tail (T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>(-1, 1)</td>
</tr>
<tr>
<td>( T )</td>
<td>(-1, 1)</td>
</tr>
</tbody>
</table>

Cause: each player tries to outguess his opponent!

Example: Best Reply

- Mixed Strategies
  - Randomized actions to avoid being outguessed
- Players’ strategies and expected payoffs
  - Players play H w.p. \( p \) and play T w.p. \( 1-p \)
  - Expected payoff of Player A
    \[
    p_A p_B + (1-p_A)(1-p_B) - p_A p_B - p_B (1-p_A) = (1 - 2 p_A) + p_B (4 p_A - 2)
    \]
  So …
  - if \( p_B > 1/2, p_A^* = 1 \) (i.e. play H)
  - if \( p_B > 1/2, p_A^* = 0 \) (i.e. play T)
  - if \( p_B = 1/2 \), then playing either H or T is equally good
Example: Nash Equilibrium

\[ p_a = \frac{1}{2} \]

Existence of Nash Equilibrium

- **Finite strategy space (J. F. Nash, 1950)**
  
  A \( n \)-player game has at least one Nash equilibrium, possibly involving mixed strategy.

- **Infinite strategy space (R.B. Rosen, 1965)**
  
  A pure-strategy Nash Equilibrium exists in a \( n \)-player concave game.

  If the payoff functions satisfy diagonally strict concavity condition, then the equilibrium is unique.

\[
(r_j - s_j) \left( \frac{\partial J_j(s_j)}{\partial s_j} \right) + (s_j - q_j) \left( \frac{\partial J_j(q_j)}{\partial q_j} \right) < 0
\]

Distributed Computation of Nash Equilibrium

- Nash equilibrium as result of “learning”
  
  - Players iteratively adjust their strategies based on locally available information
  
  - Equilibrium is reached if there is a steady state

- Two commonly used schemes

  - **Gauss-Siedel**
  
  - **Jacobian**

Convergence of Distributed Algorithms

- Algorithms may not converge for some cases

Suggested Readings

  
  - A “must-read” classic paper

  
  - Has many useful techniques

  
  - Applies game theory to routing

- And many more...

Multi-Stage Games

- **General model**
  
  - Game is played in multiple rounds
    
    - Finite or infinitely many times
    
    - Different games could be played in different rounds
      
      - Different set of actions or even players
      
      - Different solution concepts from those in static games
        
        - Analogy: optimization vs dynamic programming

- **Two special classes**
  
  - Infinitely repeated games
  
  - Stackelberg games
Infinitely Repeated Games

- **Model**
  - A single-stage game is repeated infinitely many times
  - Accumulated payoff for a player
    \[ J = \sum_{i=1}^{n} \delta^{i-1} \tau \]
    - discount factor
    - payoff from stage \( n \)
- **Main theme**: play socially more efficient moves
  - Everyone promises to play a socially efficient move in each stage
  - Punishment is used to deter “cheating”
  - Example: justice system

Cournot's Game Revisited. I

- **Cournot's Model**
  - At equilibrium each firm produces \( B/3 \), making a profit of \( B/9 \)
  - Not an “ideal” arrangement for either firm, because...
  - If a central agency decides on production quantity \( q_m \)
    \[ q_m = \arg\max (B - q) = B/2 \]
    - so each firm should produce \( B/4 \) and make a profit of \( B/8 \)
  - An aside: why \( B/4 \) is not played in the static game?
  - If firm \( A \) produces \( B/4 \), it is more profitable for firm \( B \) to produce \( 3B/8 \) than \( B/4 \)
    - Firm \( A \) then in turn produces \( 3B/16 \), and so on...

Cournot's Game Revisited. II

- **Collaboration instead of competition**
  - Question: Is it possible for two firms to reach an agreement to produce \( B/4 \) instead of \( B/3 \) each?
  - Answer: That would depend on how important future return is to each firm...
    - A firm has two choices in each round:
      - **Cooperate**: produce \( B/4 \) and make profit \( B^2/8 \)
      - **Cheat**: produce \( 3B/8 \) and make profit \( 9B^2/64 \)
    - In the subsequent rounds, cheating will cause
      - its competitor to produce \( B/3 \) as punishment
      - its own profit to drop back to \( B^2/9 \)

Cournot's Game Revisited. III

- **Is there any incentive for a firm not to cheat?**
  - Let’s look at the accumulated payoffs:
    - If it cooperates:
      \[ S_c = (1 + \delta + \delta^2 + \ldots) B^2/8 = B^2/8 (1 - \delta) \]
    - If it cheats:
      \[ S_d = 9B^2/64 + (\delta + \delta^2 + \ldots) B^2/9 = 9B^2/64 (1 - \delta) B^1 \]
  - So it will not cheat if \( S_c > S_d \). This happens only if \( \delta > 9/17 \).
- **Conclusion**
  - If future return is valuable enough to each player, then strategies exist for them to play socially efficient moves.
  - Question: What happens if player cheats in a later round?

Strategies in Repeated Games

- **A strategy**
  - is no longer a single action
  - but a complete plan of actions
  - based on possible history of plays up to current stage
  - usually includes some punishment mechanism
  - Example: in Cournot’s game, a player’s strategy is
    - produce \( B/4 \) in the first stage.
    - In the \( n \)th stage, produce \( B/4 \) if both firms have produced \( B/4 \)
    - in each of the \( n-1 \) previous stages; otherwise, produce \( B/3 \).

Equilibrium in Repeated Games

- **Subgame-perfect Nash equilibrium (SPNE)**
  - A subgame starting at stage \( n \) is
    - identical to the original infinite game
    - associated with a particular sequence of plays from the first stage to stage \( n-1 \)
  - A SPNE constitutes a Nash equilibrium in every subgame
- **Why subgame perfect?**
  - It is all about credible threats:
    - Players believe the claimed punishments indeed will be carried out by others, when it needs to be evoked.
  - So a credible threat has to be a Nash equilibrium for the subgame.
Known Results for Repeated Games

- Friedman’s Theorem (1971)
  
  Let $G$ be a single-stage game and $(e_1, \ldots, e_n)$ denote the payoff from a Nash equilibrium of $G$.
  
  If $x = (x_1, \ldots, x_n)$ is a feasible payoff from $G$ such that $x_i \geq e_i$, $\forall i$, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game of $G$ which achieves $x$ provided that discount factor $\delta$ is close enough to one.

Assignment:

Apply this theorem to Cournot’s game on an agreement other than B/4.

Suggested Readings

  
  — Friedman’s original paper
  
  — Applies repeated game to improve the efficiency of competitive routing

Stackelberg Games

- Model
  
  - One player (leader) has dominant influence over another
  
  - Typically there are two stages
  
  - One player (leader) moves first
  
  - Then the other follows in the second stage
  
  - Can be generalized to have
    - multiple groups of players
    - Static games in both stages
- Main Theme
  
  - Leader plays by backwards induction, based on the anticipated behavior of his/her follower.

Stackelberg’s Model of Duopoly

- Assumptions
  
  - Firm 1 chooses a quantity $q_1$ to produce
  
  - Firm 2 observes $q_1$ and then chooses a quantity $q_2$
- Outcome of the game
  
  - For any given $q_1$, the best move for Firm 2 is $q_2^* = (B - q_1)/2$.
  
  - Knowing this, Firm 1 chooses $q_1$ to maximize
  
    $J_1 = (B - q_1 - q_2^*)q_1 = q_1(B - q_1)/2$
  
  which yields
  
  $q_1^* = B/2$, and $q_2^* = B/4$.
  
  $J_1^* = B^2/8$, and $J_2^* = B^2/16$.

Suggested Readings

  
  — Network leads users to reach system optimal equilibrium in competitive routing.
  
  — Network charges users price to maximize its revenue.

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- Complete-Information Strategic Games
  
  - Static Games
  
  - Repeated Games
  
  - Stackelberg Games
- Cooperative Games
  
  - Nash’s Bargaining Problem
  
  - Coalitions: the Shapley value
Cooperation In Games

• Incentive to cooperate
  - Static games often lead to inefficient equilibrium
  - Achieve more efficient outcomes by acting together
    • Collusion, binding contract, side payment...

• Pareto Efficiency
  A solution is Pareto efficient if there is no other feasible solution in which some player is better off and no player is worse off.
  - Pareto efficiency may be neither socially optimal nor fair
    - Socially optimal $\Rightarrow$ Pareto efficient
    - Fairness issues

Pareto Efficiency

Reading assignment as an example

Nash's Bargaining Problem

• Model
  - Two players with interdependent payoffs $U$ and $V$
  - Acting together they can achieve a set of feasible payoffs
  - The more one player gets, the less the other is able to get
  - And there are multiple Pareto efficient payoffs

• Q: which feasible payoff would they settle on?
  - Fairness issue

Example (from Owen):
  - Two men try to decide how to split $100$
  - One is very rich, so that $U(x) \propto x$
  - The other has only $1$, so $V(x) \propto \log(1+x)$

  How would they split the money?

Intuition

• Feasible set of payoffs
  - Denote $x$ the amount that the rich man gets
    - $(u,v)=(x, \log(101-x))$, $x \in [0,100]$
  - Find the allocation which maximizes $DuV$

  A fair split should satisfy
  \[
  \frac{\Delta u}{u} = \frac{\Delta v}{v}
  \]

Let $a \to 0$, $\frac{du}{a} = -\frac{dv}{a}$
Or $\frac{du}{a} + \frac{dv}{a} = 0$, or $\frac{d}{a}u = 0$, or $\frac{d}{a}v = 0$
\[
\Rightarrow \text{Find the allocation which maximizes } DuV
\]
\[
\Rightarrow x^*=76.8!
\]

Results

• There is a unique solution which
  - satisfies the above axioms
  - maximizes the product of two players' additional payoffs $(u-u_0)(v-v_0)$
  - This solution can be enforced by “threats”
  - Each player independently announces his/her threat
  - Players then bargain on their threats
  - If they reach an agreement, that agreement takes effect;
    - Otherwise, initially announced threats will be used
  - Different fairness criteria can be achieved by changing the last axiom (see references)

Nash's Axiomatic Approach (1950)

• A solution $(u^*, v^*)$ should be
  - Rational
    - $(u, v) \not\in S$, where $(u_0, v_0)$ is the worst payoffs that the players can get.
  - Feasible
    - $(u, v) \in S$, the set of feasible payoffs.
  - Pareto efficient
  - Symmetric
    - If $S$ is such that $(u, v) \in S \Rightarrow (v, u) \in S$, then $u^*=v^*$.
  - Independent from linear transformations
  - Independent from irrelevant alternatives
    - Suppose $T \subset S$. If $(u, v) \in T$ is a solution to $S$, then $(u, v)$ should also be a solution to $T$

Suggested Readings

  - Nash’s original paper. Very well written.
• X. Cao. “Preference Functions and Bargaining Solutions.”
  - A paper which unifies all bargaining solutions into a single framework
  - Applies Nash’s bargaining solution to resource allocation problem in admission control (multi-objective optimization)
Coalitions

- Model
  - Players (n>2) N form coalitions among themselves
  - A coalition is any nonempty subset of N
- Characteristic function V defines a game
  \[ V(S) = \text{payoff to S in the game between S and N-S}, \forall S \subset N \]
  \[ V(N) = \text{total payoff achieved by all players acting together} \]
- Assumed to be super-additive
  \[ V(S+T) \geq V(S)+V(T), \forall S, T \subset N \]

Questions of Interest

- Condition for forming stable coalitions
- When will a single coalition be formed?
- How to distribute payoffs among players in a fair way?

Core Sets

- Allocation \( X=(x_1, \ldots, x_n) \)
  \[ x_i \geq V({i}), \forall i \in N; \sum_{i \in N} x_i = V(N) \]

The core of a game

- A set of allocation which satisfies \( \sum_{i \in S} x_i \geq V(S), \forall S \subset N \)
- If the core is nonempty, a single coalition can be formed

An example

- A Berkeley landlord (L) is renting out a room
- Al (A) and Bob (B) are willing to rent the room at $600 and $800, respectively
- Who should get the room at what rent?

Example: Core Set

- Characteristic function of the game
  - \( V(L)=V(A)=V(B)=V(A+B)=0 \)
  - Coalition between L and A or L and B
    - if rent = x, then L’s payoff = x, A’s payoff = 600 – x
    - so \( V(L+A) = 600 \).
    - Similarly, \( V(L+B) = 800 \).
  - Coalition among L, A and B: \( V(L+A+B) = 800 \)
- The core of the game
  \[ x_L+x_A \geq 600 \]
  \[ x_L+x_B \geq 800 \]
  \[ x_L+x_A+x_B = 800 \]
  \[ \Rightarrow \text{core} = \{(y,0,800-y), 600 \leq y \leq 800\} \]

Fair Allocation: the Shapley Value

- Define solution for player \( i \) in game \( V \) by \( P_i(V) \)
- Shapley’s axioms
  - \( P_i \)'s are independent from permutation of labels
  - Additive: if \( U \) and \( V \) are any two games, then
    \[ P_i(U+V) = P_i(U) + P_i(V), \forall i \in N \]
  - \( T \) is a carrier of \( N \) if \( V(S \cap T) = V(S), \forall S \subset N \).
    Then for any carrier \( T, \sum_{i \in T} P_i = V(T) \).
- Unique solution: Shapley’s value (1953)
  \[ P_i = \sum_{S \subset N} (|S|-1)! \frac{(|N|-|S|)!}{N!} \left[ V(S) - V(S \setminus \{i\}) \right] \]
- Intuition: a probabilistic interpretation

Suggested Readings

  - Shapley’s original paper.
  - Applies Shapley’s value to caller-ID service.
  - How coalition could improve the revenue of international telephone carriers.

Summary

- Models
  - Strategic games
    - Static games, multi-stage games
  - Cooperative games
    - Bargaining problem, coalitions
- Solution concepts
  - Strategic games
    - Nash equilibrium, Subgame-perfect Nash equilibrium
  - Cooperative games
    - Nash’s solution, Shapley value
- Application to networking research
  - Modeling and design
References

  — an easy-to-read introductory to the subject
  — a concise but rigorous treatment on the subject
  — a good reference on cooperative games
  — a complete handbook; “the bible for game theory”  