Image Reconstruction from Projection

- Reconstruct an image from a series of projections

  X-ray computed tomography (CT)

"Computed tomography is a medical imaging method employing tomography where digital geometry processing is used to generate a three-dimensional image of the internals of an object from a large series of two-dimensional X-ray images taken around a single axis of rotation."

http://en.wikipedia.org/wiki/Computed_tomography

Backprojection

" In computed tomography or other imaging techniques requiring reconstruction from multiple projections, an algorithm for calculating the contribution of each voxel of the structure to the measured ray data, to generate an image; the oldest and simplest method of image reconstruction. "

http://www.medilexicon.com/medicaldictionary.php?t=9165

Image Reconstruction: Introduction

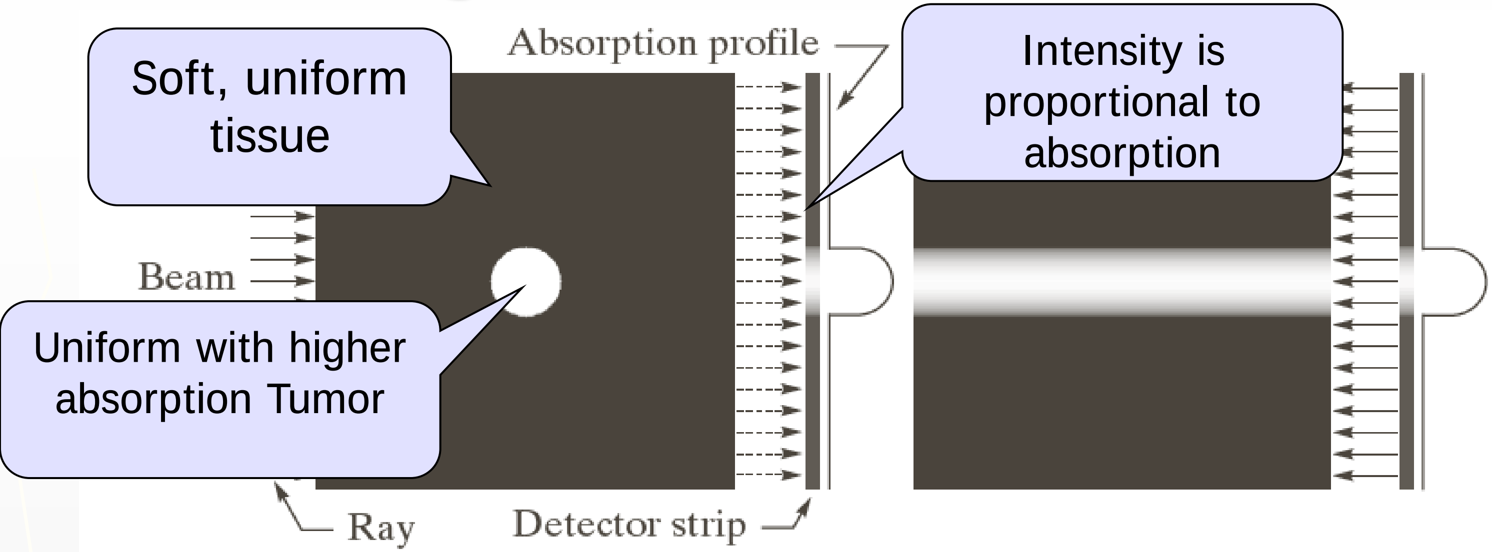
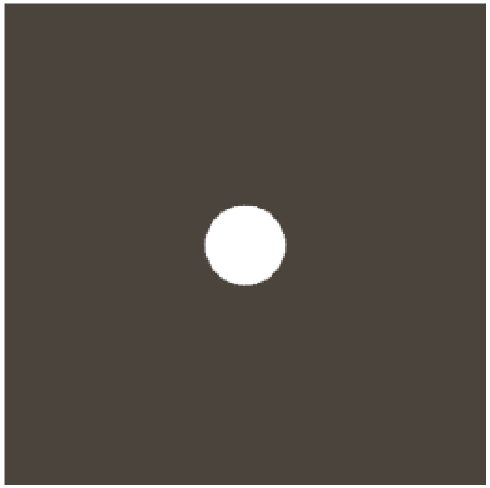


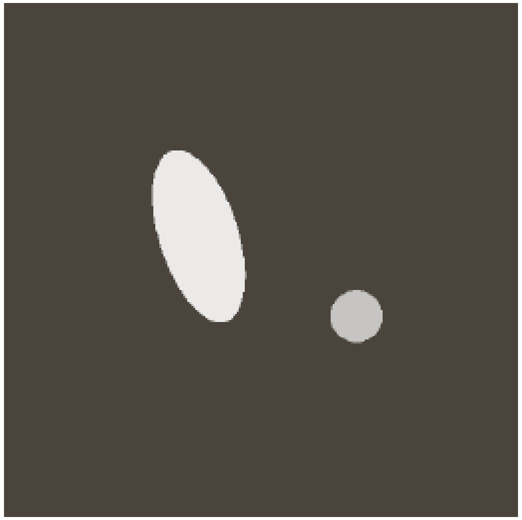
FIGURE 5.32
(a) Flat region showing a simple object, an input parallel beam, and a detector strip. (b) Result of back-projecting the sensed strip data (i.e., the 1-D absorption profile). (c) The beam and detectors rotated by 90°. (d) Back-projection. (e) The sum of (b) and (d). The intensity where the back-projections intersect is twice the intensity of the individual back-projections.

Image Reconstruction: Introduction

a	b	c
d	e	f

FIGURE 5.33
(a) Same as Fig. 5.32(a).
(b)–(e) Reconstruction using 1, 2, 3, and 4 backprojections 45° apart.
(f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).





a	b	c
d	e	f

FIGURE 5.34 (a) A region with two objects. (b)–(d) Reconstruction using 1, 2, and 4 backprojections 45° apart. (e) Reconstruction with 32 backprojections 5.625° apart. (f) Reconstruction with 64 backprojections 2.8125° apart.

a b
c d

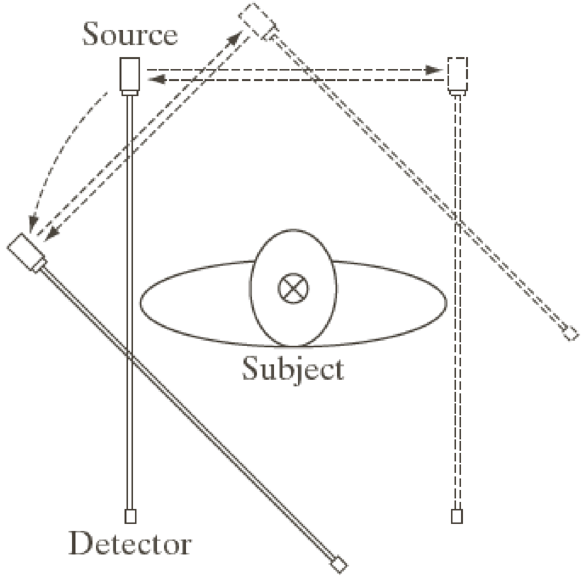


FIGURE 5.35 Four generations of CT scanners. The dotted arrow lines indicate incremental linear motion. The dotted arrow arcs indicate incremental rotation. The cross-mark on the subject's head indicates linear motion perpendicular to the plane of the paper. The double arrows in (a) and (b) indicate that the source/detector unit is translated and then brought back into its original position.

Other CTs

- **Electron beam CT** (Fifth-generation CT)

  Electron beam tomography (EBCT) was introduced in the early 1980s, by medical physicist Andrew Castagnini, as a method of improving the temporal resolution of CT scanners.

  High cost of EBCT equipment, and poor flexibility

- **Helical (or spiral) cone beam computed tomography** (sixth-generation)

  A type of three dimensional computed tomography (CT) in which the source (usually of x-rays) describes a helical trajectory relative to the object while a two dimensional array of detectors measures the transmitted radiation on part of a cone of rays emitting from the source

http://en.wikipedia.org/wiki/Computed_tomography

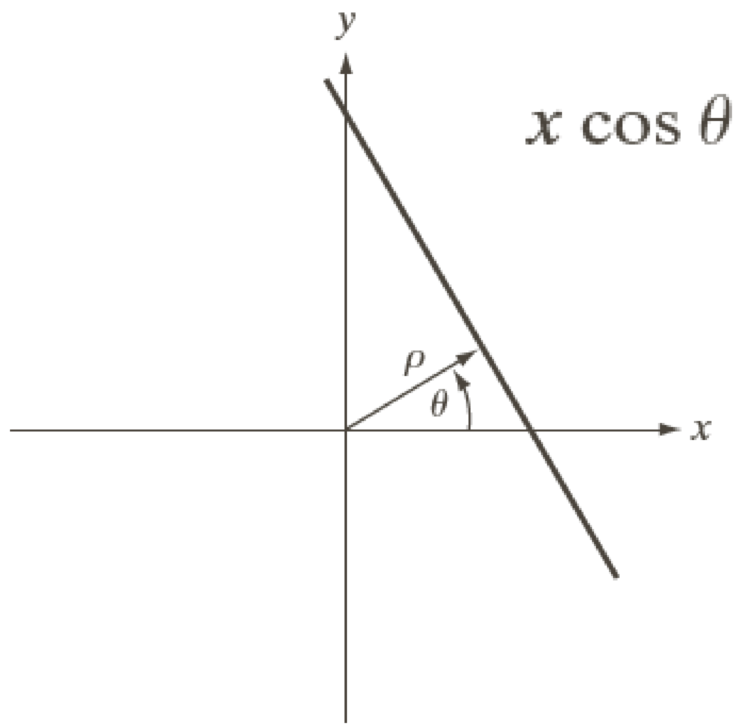
Other CTs

- Multislice CT (seventh-generation)

- The major benefit of multi-slice CT

  - Significant increase in detail
  - Utilizes X-ray tubes more economically
  - Reducing cost and potentially reducing dosage

Projections and the Radon Transform

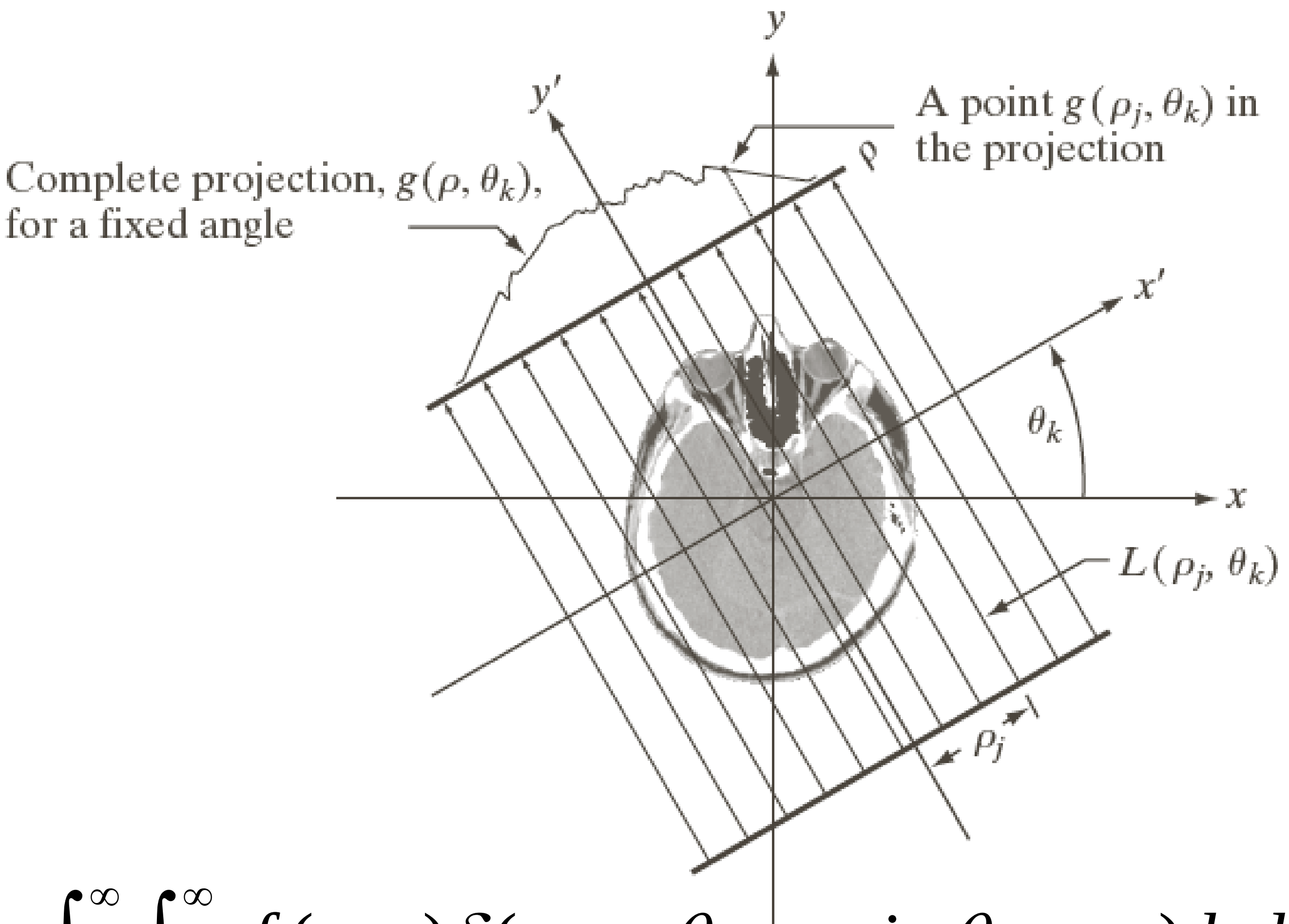


$$x \cos\theta + y \sin\theta = \rho$$

FIGURE 5.36 Normal representation of a straight line.

Projections and the Radon Transform

FIGURE 5.37 Geometry of a parallel-ray beam.



$$g(\rho_j, \theta_k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x\cos\theta_k + y\sin\theta_k - \rho_j)dxdy$$

Projections and the Radon Transform

- **Radon transform** gives the projection (line integral) of f(x,y) along an arbitrary line in the xy-plane

$$\Re\{f\} = g(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)dxdy$$

$$\Re\{f\} = g(\rho,\theta) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)$$

Example: Using the Radon transform to obtain the projection of a circular region

- Assume that the circle is centered on the origin of the xy-plane. Because the object is circularly symmetric, its projections are the same for all angles, so we just check the projection for $\theta = 0^\circ$

$$f(x, y) = \begin{cases} A & x^2 + y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Example: Using the Radon transform to obtain the projection of a circular region

$$g(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)dxdy$$

$$g(\rho) = \begin{cases} 2A\sqrt{r^2 - \rho^2} & |\rho| \leq \text{r} \\ 0 & \text{otherwise} \end{cases}$$

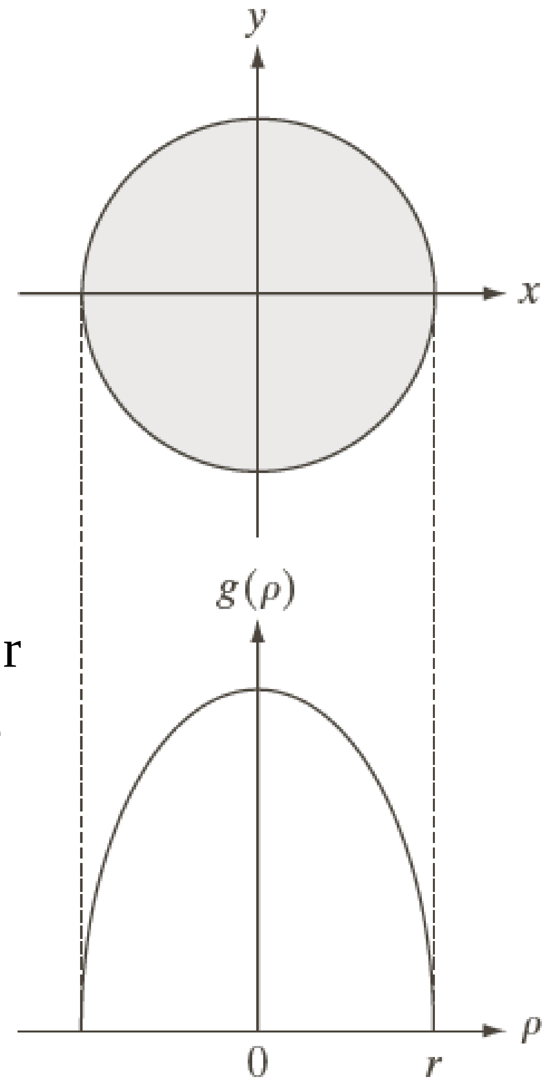


FIGURE 5.38 A disk and a plot of its Radon transform, derived analytically. Here we were able to plot the transform because it depends only on one variable. When g depends on both ρ and θ, the Radon transform becomes an image whose axes are ρ and θ, and the intensity of a pixel is proportional to the value of g at the location of that pixel.

Sinogram: The Result of Radon Transform

- Sinogram: the result of Radon transform is displayed as an image with ρ and θ as rectilinear coordinates

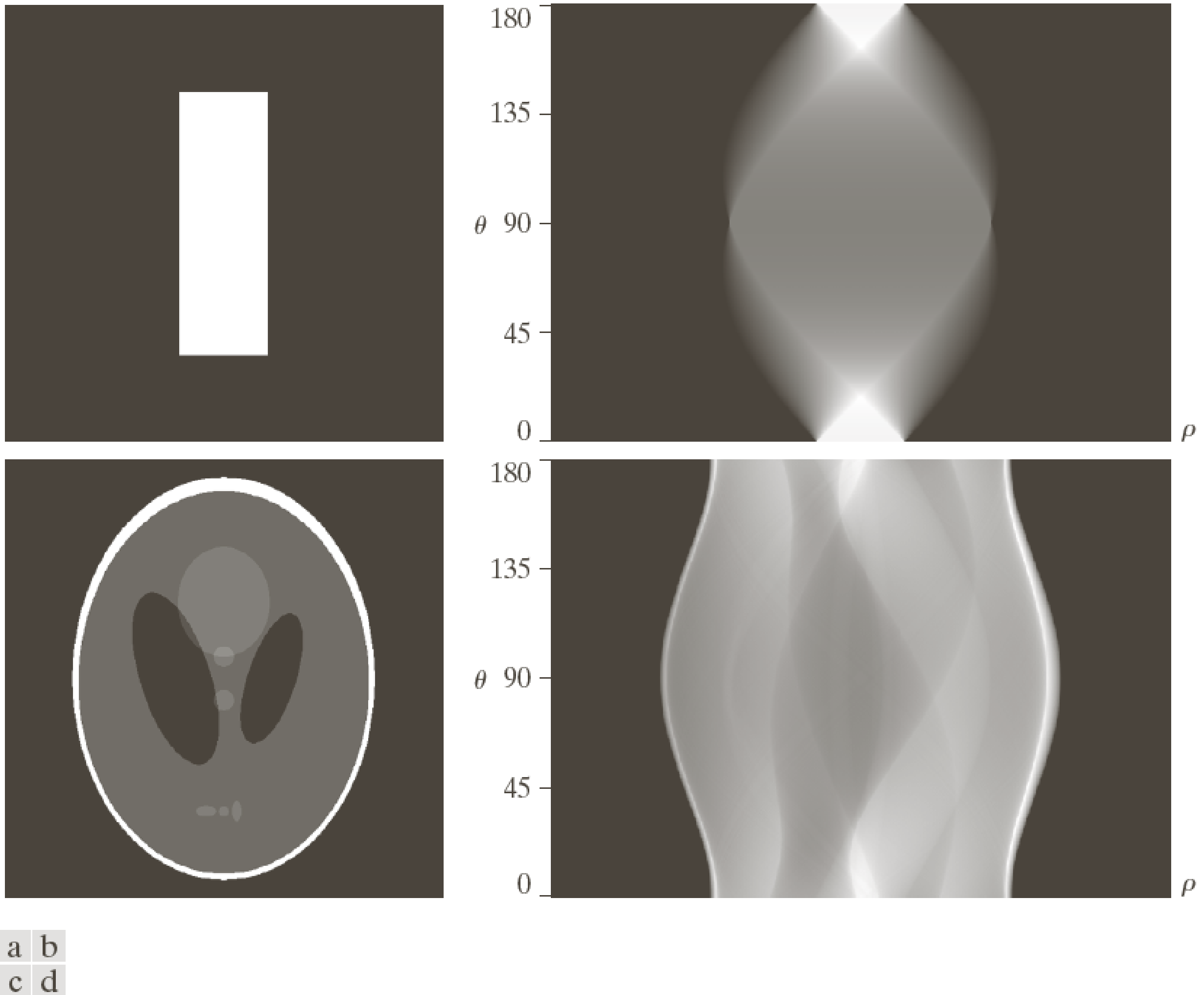


FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

Image Reconstruction

$$f_\theta(x,y) = g(x\cos\theta + y\sin\theta, \theta)$$

$$f(x,y) = \int_0^\pi f_\theta(x,y)\,d\theta$$

$$f(x,y) = \sum_{\theta=0}^{\pi} f_\theta(x,y)$$

A back-projected image formed is referred to as a laminogram

Examples: Laminogram

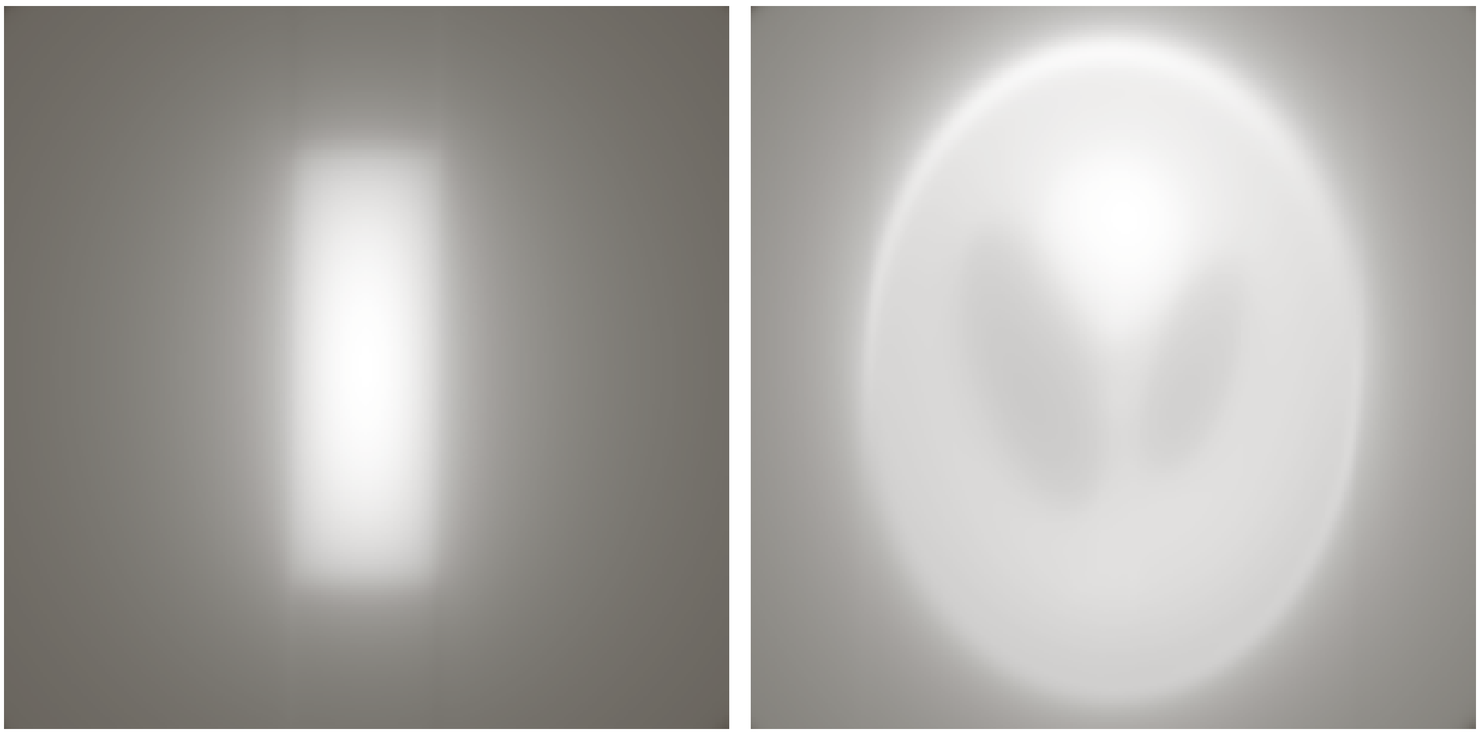


FIGURE 5.40 Backprojections of the sinograms in Fig. 5.39.

The Fourier-Slice Theorem

For a given value of θ, the 1-D Fourier transform of a projection with respect to ρ is

$$G(w,\theta) = \int_{-\infty}^{\infty} g(\rho,\theta) e^{-j2\pi\omega\rho} d\rho$$

$$G(\omega,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho) e^{-j2\pi\omega\rho} d\rho dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left[\int_{-\infty}^{\infty}\delta(x\cos\theta + y\sin\theta - \rho) e^{-j2\pi\omega\rho} d\rho\right] dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\omega(x\cos\theta + y\sin\theta)} dx dy$$

The Fourier-Slice Theorem

$$G(w,\theta) = \int_{-\infty}^{\infty} g(\rho,\theta) e^{-j2\pi\omega\rho} d\rho$$

$$G(\omega,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi\omega(x\cos\theta + y\sin\theta)} dxdy$$

$$= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy\right]_{u=w\cos\theta, v=w\sin\theta}$$

$$= \left[F(u,v)\right]_{u=w\cos\theta, v=w\sin\theta}$$

$$= F(w\cos\theta, w\sin\theta)$$

<u>Fourier-slice theorem</u>: The Fourier tansform of a projection is a slice of the 2-D Fourier transform of the region from which the projection was obtained

Illustration of the Fourier-slice theorem

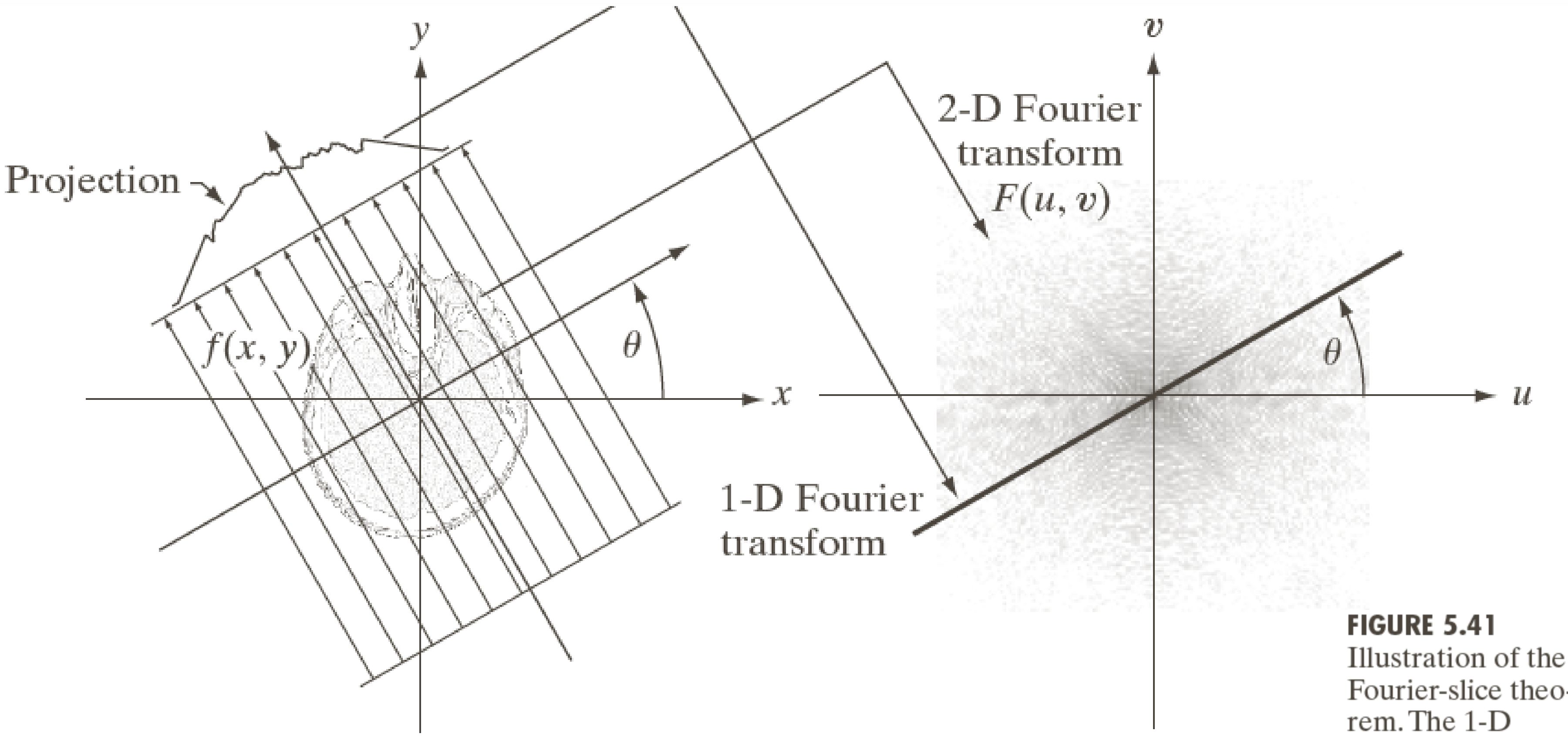


FIGURE 5.41
Illustration of the Fourier-slice theorem. The 1-D Fourier transform of a projection is a slice of the 2-D Fourier transform of the region from which the projection was obtained. Note the correspondence of the angle θ.

Reconstruction Using Parallel-Beam Filtered Backprojections

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

Let $u = w\cos\theta, v = w\sin\theta,$ then $dudv = wdwd\theta,$

$$f(x,y)=\int_{0}^{2\pi}\int_{0}^{\infty}F(w\cos\theta, w\sin\theta)e^{j2\pi w(x\cos\theta+y\sin\theta)}wdwd\theta$$

$$=\int_{0}^{2\pi}\int_{0}^{\infty}G(w,\theta)e^{j2\pi w(x\cos\theta+y\sin\theta)}wdwd\theta$$

$$G(w,\theta+180^\circ)=G(-w,\theta)$$

$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{\infty}|w|G(w,\theta)e^{j2\pi w(x\cos\theta+y\sin\theta)}dwd\theta$$

Reconstruction Using Parallel-Beam Filtered Backprojections

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{\infty} |w| G(w,\theta) e^{j2\pi w(x\cos\theta + y\sin\theta)} dw d\theta$$

$$= \int_0^{\pi} \left[\int_{-\infty}^{\infty} |w| G(w,\theta) e^{j2\pi w\rho} dw \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

It's not integrable

Approach:

Window the ramp so it becomes zero outside of a defined frequency interval. That is, a window band-limits the ramp filter.

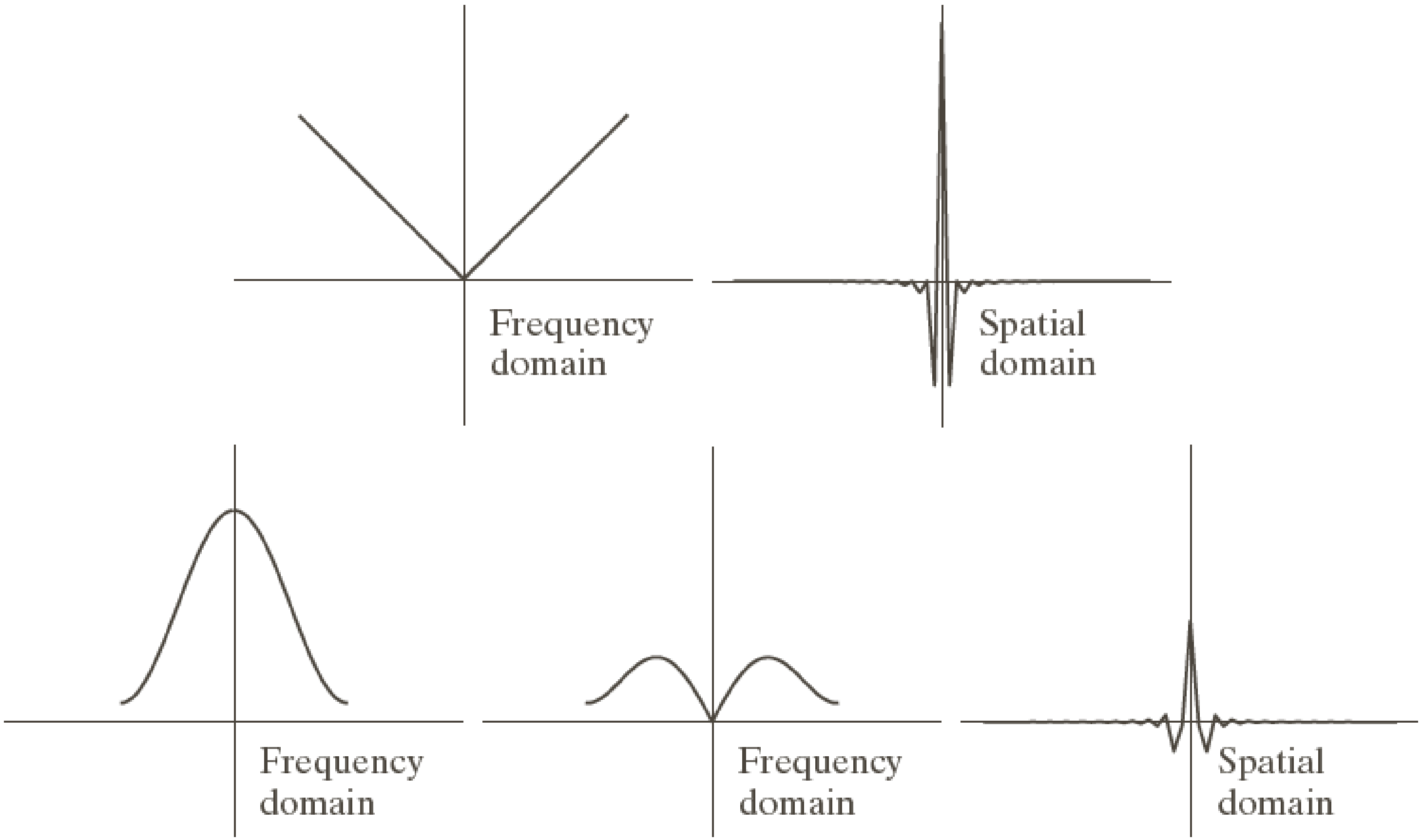
Hamming / Hann Widow

$$h(w) = \begin{cases} c + (c-1)\cos\dfrac{2\pi w}{M-1} & 0 \le w \le (M-1) \\ 0 & \text{otherwise} \end{cases}$$

$c = 0.54,$ the function is called the <u>Hamming window</u>

$c = 0.5,$ the function is called the <u>Han window</u>

The Plot of Hamming Widow



a b
c d e

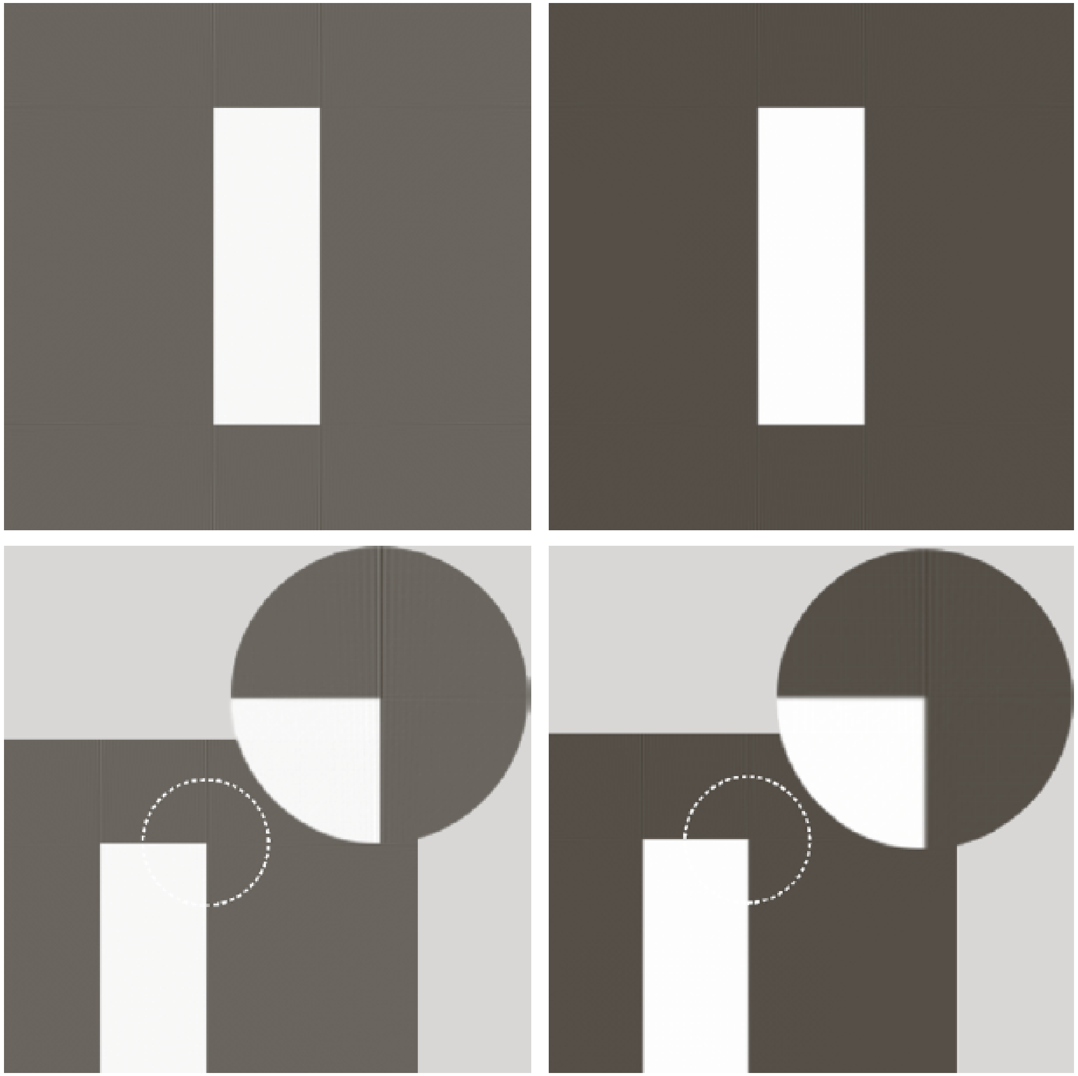
FIGURE 5.42
(a) Frequency domain plot of the filter $|\omega|$ after band-limiting it with a box filter. (b) Spatial domain representation. (c) Hamming windowing function. (d) Windowed ramp filter, formed as the product of (a) and (c). (e) Spatial representation of the product (note the decrease in ringing).

Filtered Backprojection

The complete, filtered backprojection (to obtain the reconstructed image f(x,y)) is described as follows:

1. Compute the 1-D Fourier transform of each projection
2. Multiply each Fourier transform by the filter function |w| which has been multiplied by a suitable (e.g., Hamming) window
3. Obtain the inverse 1-D Fourier transform of each resulting filtered transform
4. Integrate (sum) all the 1-D inverse transforms from step 3

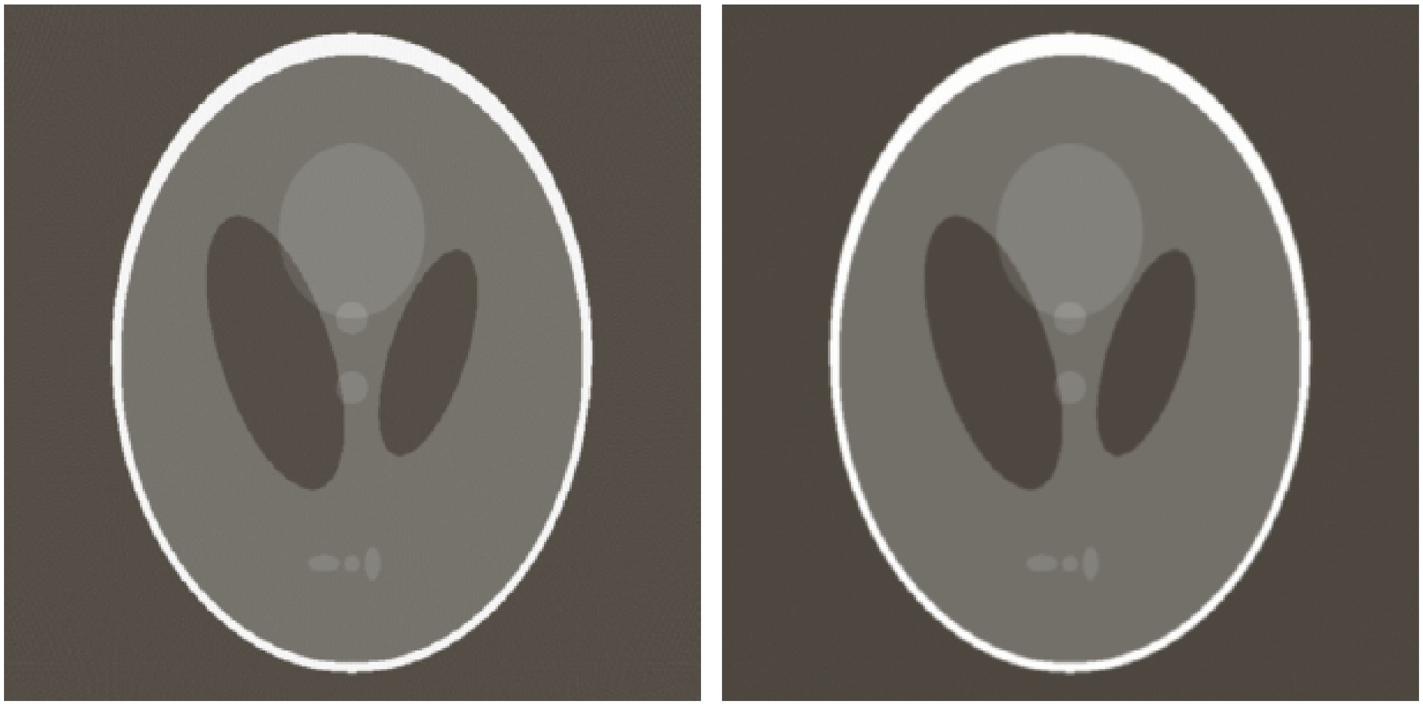
Examples: Filtered Backprojection



a	b
c	d

FIGURE 5.43 Filtered back-projections of the rectangle using (a) a ramp filter, and (b) a Hamming-windowed ramp filter. The second row shows zoomed details of the images in the first row. Compare with Fig. 5.40(a).

Examples: Filtered Backprojection



a b

FIGURE 5.44 Filtered backprojections of the head phantom using (a) a ramp filter, and (b) a Hamming-windowed ramp filter. Compare with Fig. 5.40(b).

Implementation of Filtered Backprojection in Spatial Domain

- Fourier transform of the product of two frequency domain functions is equal to the convolution of the spatial representation
- Let s(p) denote the inverse Fourier transform of |w|

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} |w| G(w,\theta) e^{j2\pi w\rho} dw \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

$$= \int_0^{\pi} \left[s(\rho) \bigstar g(\rho,\theta) \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

$$= \int_0^{\pi} \left[\int_{-\infty}^{\infty} g(\rho,\theta) s(x\cos\theta + y\sin\theta - \rho) d\rho \right] d\theta$$

Reconstruction Using Fan-Beam Filtered Backprojections

$$\theta = \alpha + \beta$$

$$\rho = D \sin \alpha$$

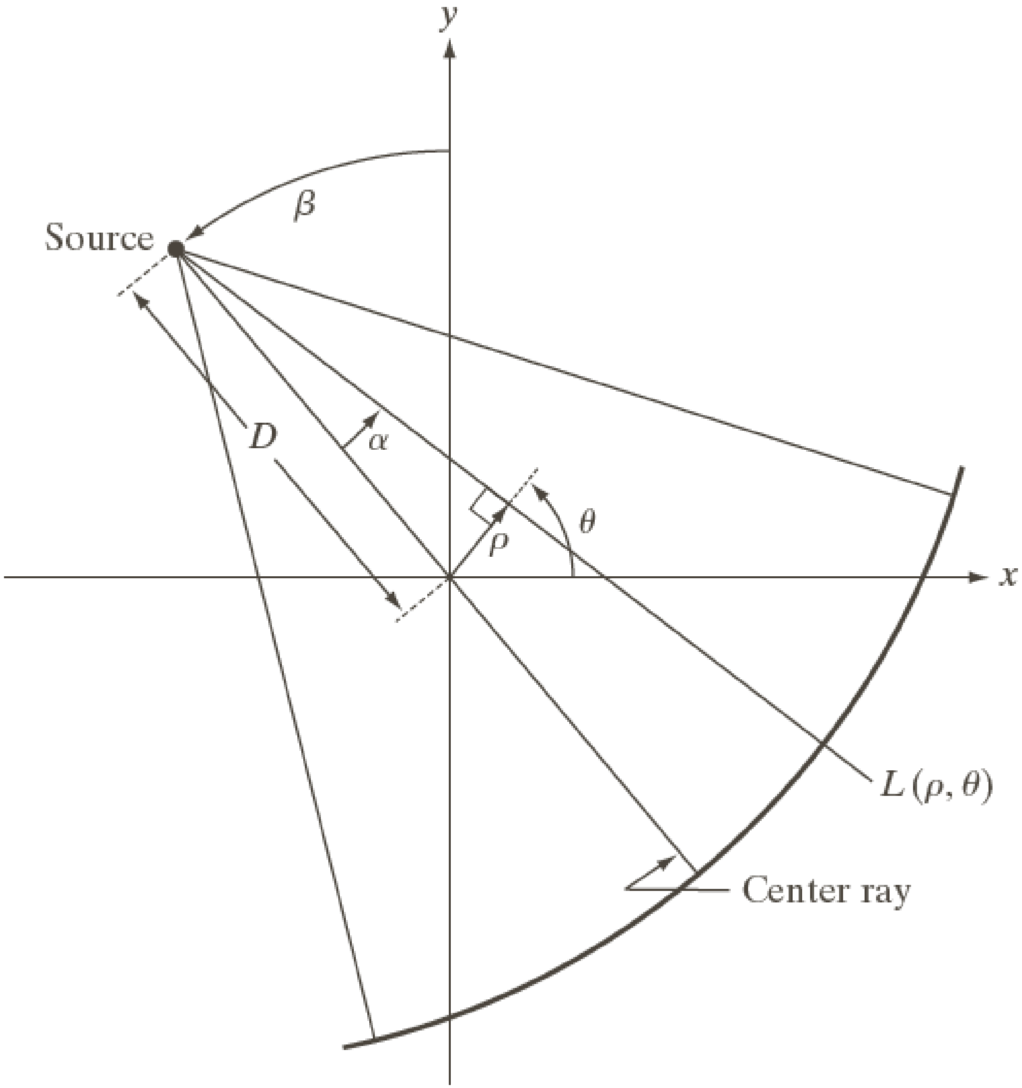


FIGURE 5.45
Basic fan-beam geometry. The line passing through the center of the source and the origin (assumed here to be the center of rotation of the source) is called the *center ray*.

Reconstruction Using Fan-Beam Filtered Backprojections

Objects are encompassed within a circular area of radius T about the origin of the plane, or g(ρ,θ)=0 for |ρ|>T

$$f(x,y)=\int_0^{\pi}\left[\int_{-\infty}^{\infty} g(\rho,\theta)s(x\cos\theta+y\sin\theta-\rho)d\rho\right]d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)s(x\cos\theta+y\sin\theta-\rho)d\rho d\theta$$

$$x=r\cos\varphi;\, y=r\sin\varphi$$

$$x\cos\theta+y\sin\theta=r\cos\varphi\cos\theta+r\sin\varphi\sin\theta$$

$$=r\cos(\varphi-\theta)$$

Reconstruction Using Fan-Beam Filtered Backprojections

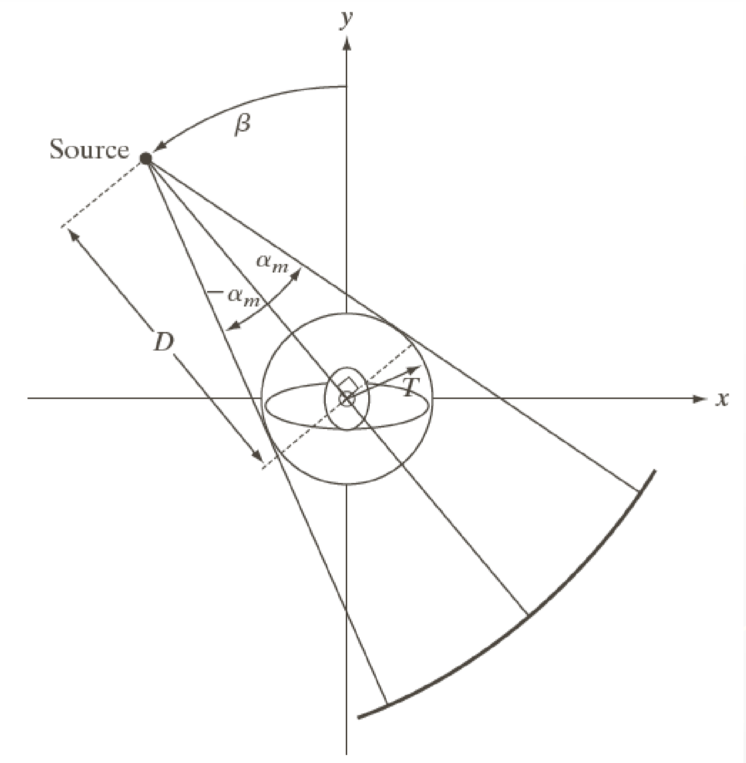
$$x = r\cos\varphi; y = r\sin\varphi$$

$$x\cos\theta + y\sin\theta = r\cos\varphi\cos\theta + r\sin\varphi\sin\theta$$

$$= r\cos(\varphi - \theta)$$

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)s\left[r\cos(\varphi-\theta)-\rho\right]d\rho d\theta$$

Reconstruction Using Fan-Beam Filtered Backprojections



$$d\rho d\theta = D\cos\alpha \, d\alpha \, d\beta$$

$$f(x,y) = \frac{1}{2}\int_{0}^{2\pi}\int_{-T}^{T} g(\rho,\theta)s\left[r\cos(\varphi-\theta)-\rho\right]d\rho d\theta$$

$$= \frac{1}{2}\int_{-\alpha}^{2\pi-\alpha}\int_{-\sin^{-1}(-T/D)}^{\sin^{-1}(T/D)} g(D\sin\alpha,\alpha+\beta)s\left[r\cos(\alpha+\beta-\varphi)-D\sin\alpha\right]D\cos\alpha \, d\alpha \, d\beta$$

Reconstruction Using Fan-Beam Filtered Backprojections

$$f(x,y)=\frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)s\left[r\cos(\varphi-\theta)-\rho\right]d\rho d\theta$$

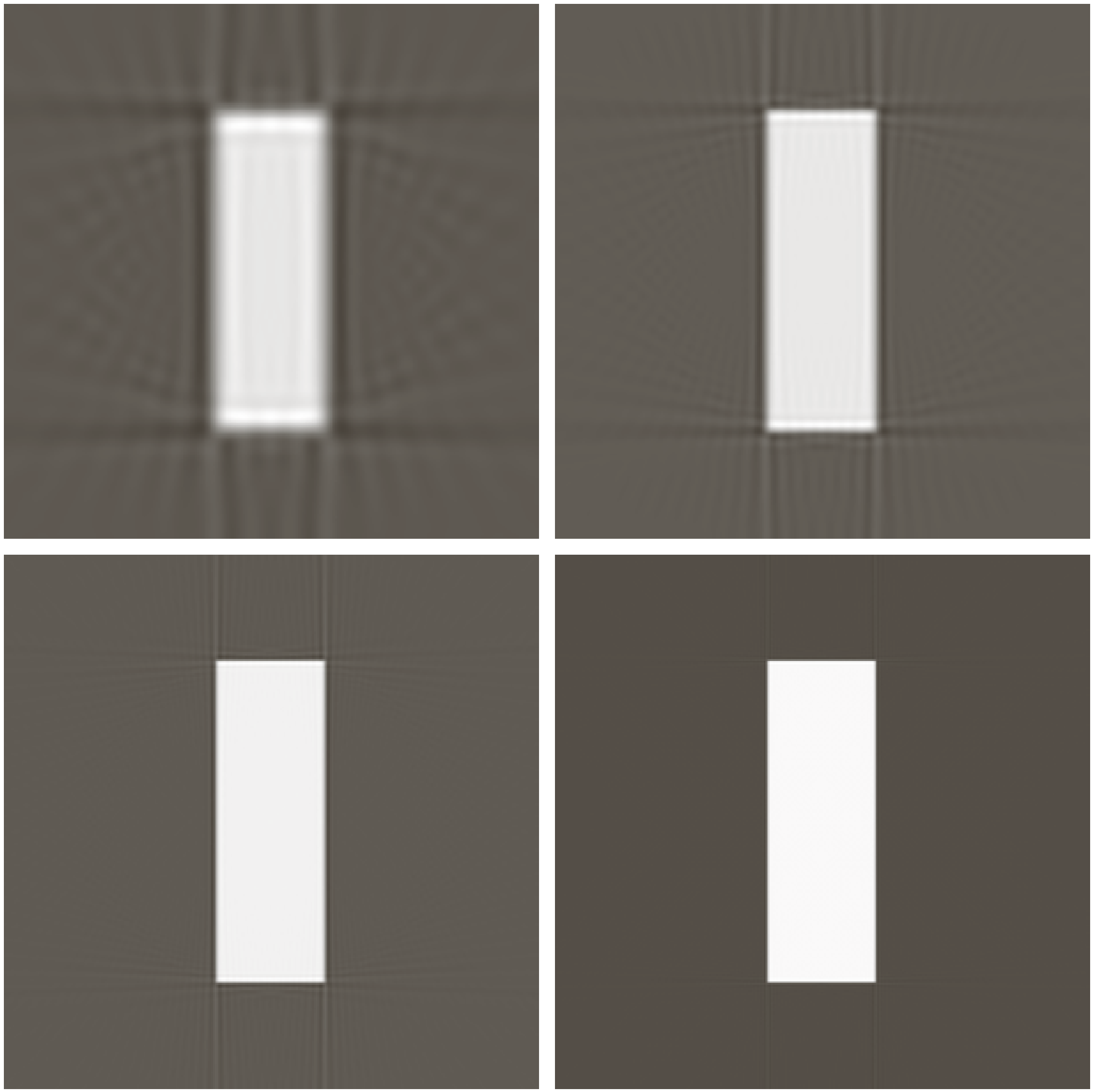
$$=\frac{1}{2}\int_{-\alpha}^{2\pi-\alpha}\int_{-\sin^{-1}(-T/D)}^{\sin^{-1}(T/D)} g(D\sin\alpha,\alpha+\beta)s\left[r\cos(\alpha+\beta-\varphi)-D\sin\alpha\right]D\cos\alpha\, d\alpha\, d\beta$$

$$f(r,\varphi)=\frac{1}{2}\int_0^{2\pi}\int_{-\alpha_m}^{\alpha_m} p(\alpha,\beta)s\left[R\sin\left(\alpha'-\alpha\right)\right]D\cos\alpha\, d\alpha\, d\beta$$

$$s(R\sin\alpha)=\left(\frac{\alpha}{R\sin\alpha}\right)^2 s(\alpha)$$

$$f(r,\varphi)=\int_0^{2\pi}\frac{1}{R^2}\left[\int_{-\alpha_m}^{\alpha_m} q(\alpha,\beta)h\left(\alpha'-\alpha\right)d\alpha\right]d\beta$$

$$h(\alpha)=\frac{1}{2}\left(\frac{\alpha}{\sin\alpha}\right)^2 s(\alpha),\quad q(\alpha,\beta)=p(\alpha,\beta)D\cos\alpha$$



a b
c d

FIGURE 5.48
Reconstruction of the rectangle image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.43(b).

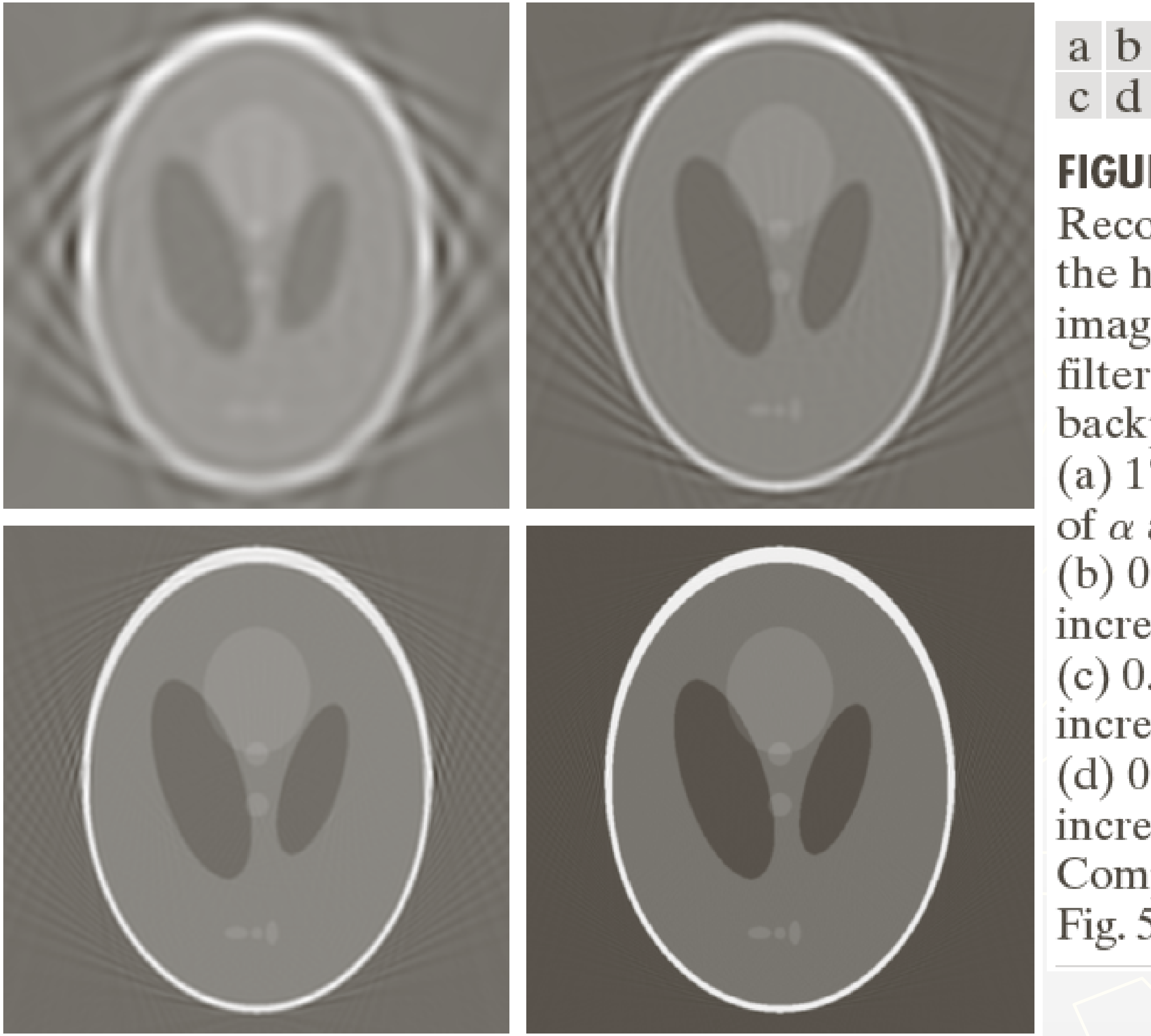


FIGURE 5.49 Reconstruction of the head phantom image from filtered fan backprojections. (a) 1° increments of α and β. (b) 0.5° increments. (c) 0.25° increments. (d) 0.125° increments. Compare (d) with Fig. 5.44(b).