Mathematical Aspects/Derivation of Histogram Equalization

- Consider continuous values of intensity. rather than discretized.

- r = gray level of image to be equalized

  r normalized to $[0, 1]$

  $0 \rightarrow$ dark = Black

  $1 \rightarrow$ Bright = White.

- Goal: Design a Transformation.

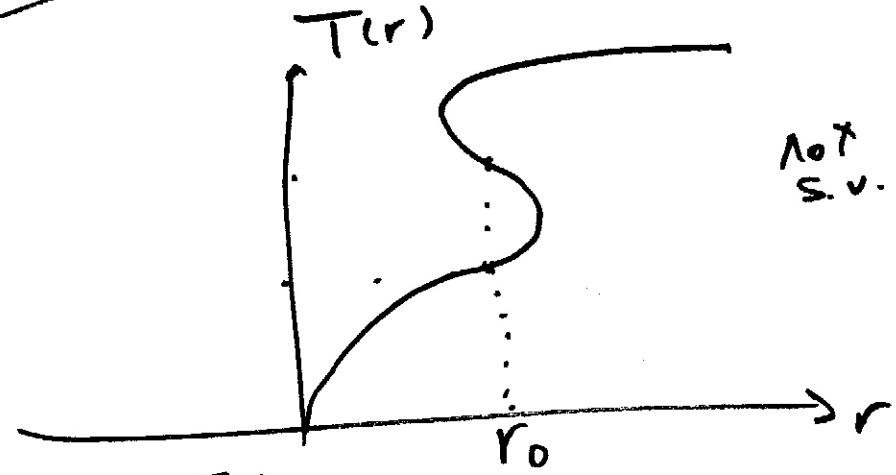
histogram matching or equalization $\longleftarrow$ $T(r) = S$ $\quad 0 \leq r \leq 1$

$0 \leq S \leq 1$

- Assumptions about $T(r)$.

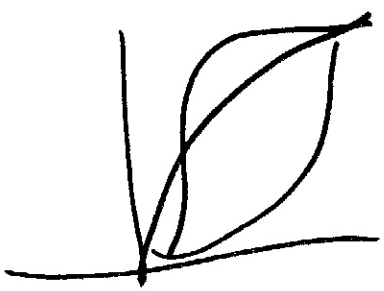
(a) $T(r)$ <u>single valued</u> and <u>monotically increasing</u> in the interval $0 \le r \le 1$

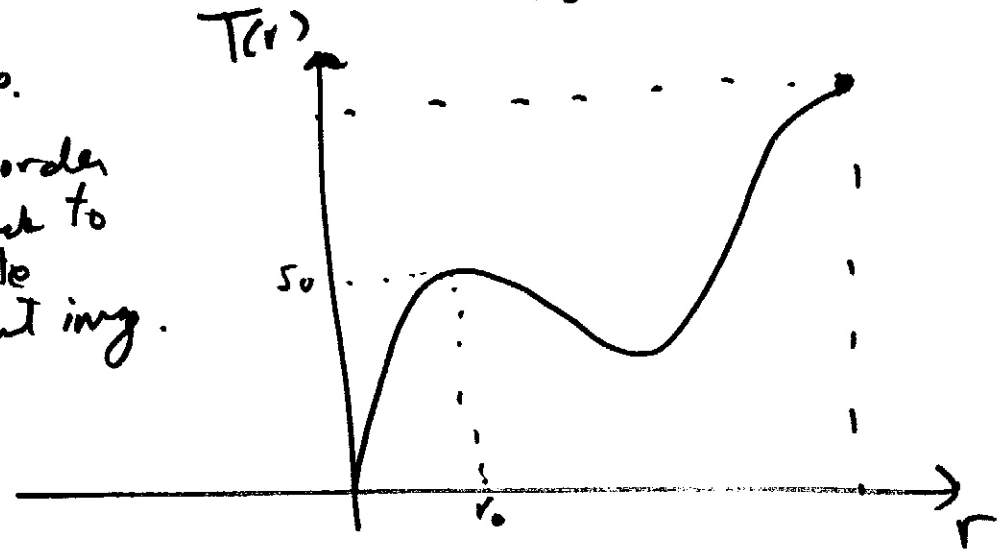
→ for inverse transformation to exist



Not s.v.

→ preserve increasing order from black to the white in output img.





(b) $0 \leq T(r) \leq 1$ for $0 \leq r \leq 1$

$\rightarrow$ output grey levels in the same range as input levels.

$T(r) = s$

$T^{-1}(s) = r$

r = input intensity
s = output intensity.

<u>Result from prob theory</u>:

If $T^{-1}(s)$ satisfies condition (a) then.

$$P_s(s) = P_r(r) \left| \frac{dr}{ds} \right|$$

Consider CDF: as a Transformation.

$$s = T(r) = \int_0^r p_r(w)\,dw$$

$$\frac{ds}{dr} = \frac{d}{dr}\left\{\int_0^r p_r(w)\,dw\right\} = p_r(r)$$

$$p_s(s) = p_r(r)\left|\frac{1}{p_r(r)}\right| = 1 \qquad 0 \le s \le 1$$

<u>Words</u>: If $T(r)$ is just a CDF or just the integral of input pdf $(p_r(r))$ Then ~~and~~ applying $T(r)$ results in a image whose pdf $(p_s(s))$ is Uniform

Discrete Case

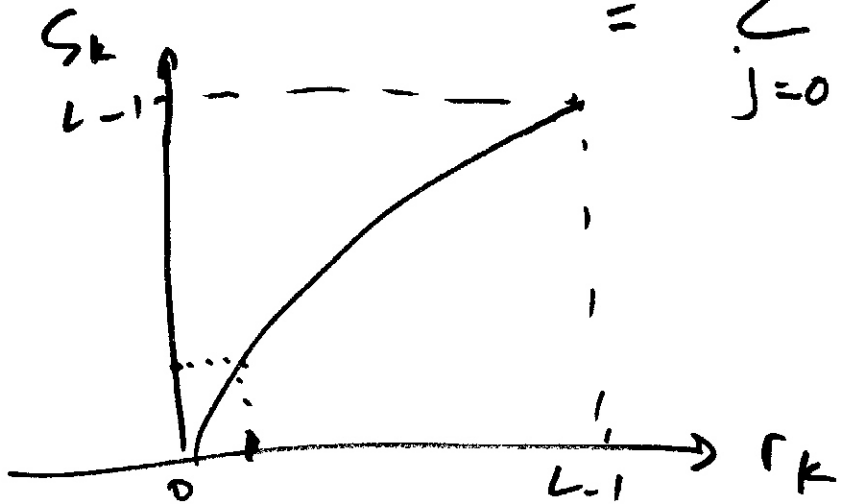
r_k = discrete intensity value. $k = 0, \ldots, L-1$

$$p_r(r_k) = \frac{n_k}{n} \qquad k = 0, \ldots, L-1$$

n_k = # of pixels that have intensity r_k.

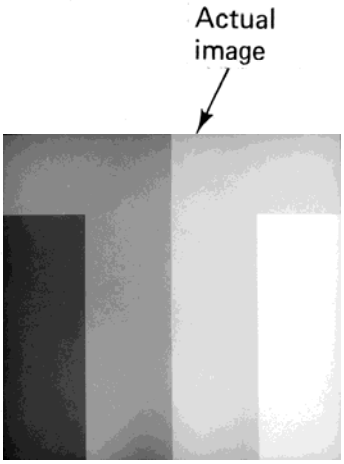
$$s_k = T(r_k) = \sum_{j=0}^{k} p_r(r_j) \longleftarrow$$

$$= \sum_{j=0}^{k} \frac{n_j}{n} \qquad k = 0, \ldots, L-1$$

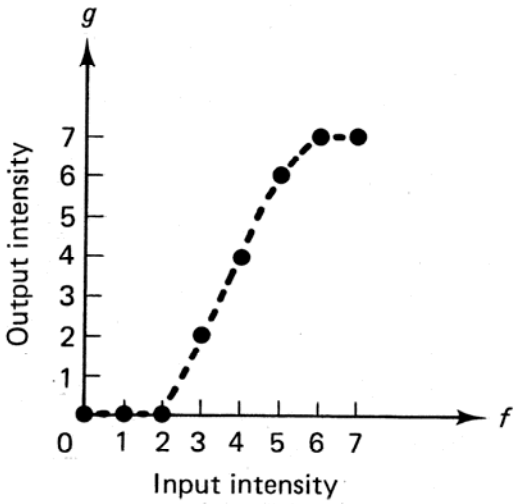


Two-Dimensional Signal and Image Processing
Chapter 8 - pictures

3	3	4	4
2	3	4	5
2	3	4	5
2	3	4	5



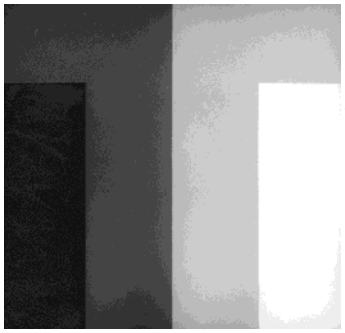
(a)



f	g
0	0
1	0
2	0
3	2
4	4
5	6
6	7
7	7

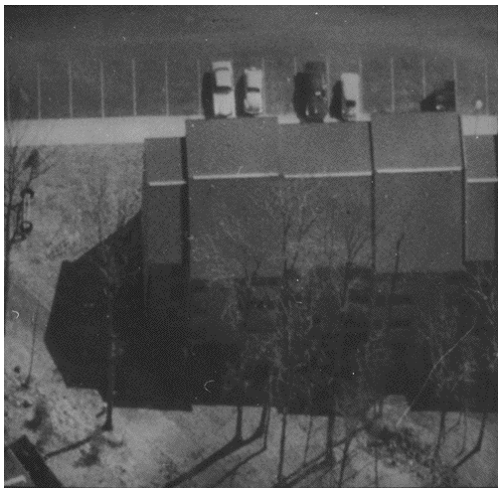
(b)

2	2	4	4
0	2	4	6
0	2	4	6
0	2	4	6

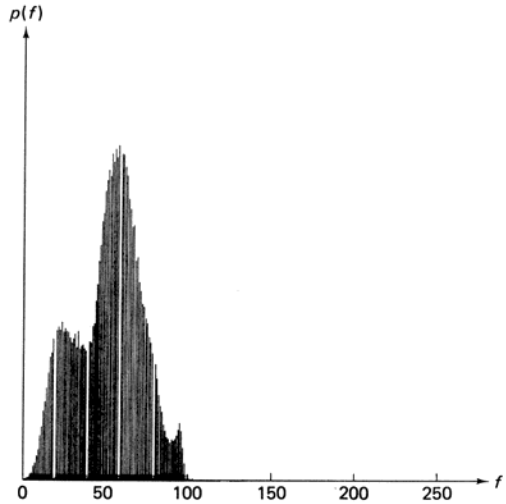


(c)

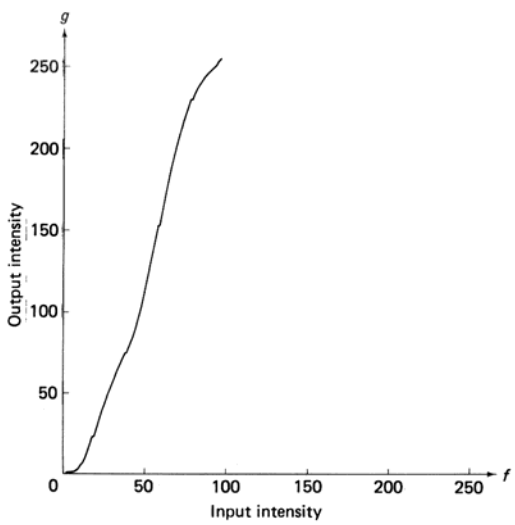
Figure 8.1 Example of gray scale modification. (a) Image of 4 x 4 pixels, with each pixel represented by 3 bits; (b) gray scale transformation function; (c) result of modifying the image in (a) using the gray scale transformation function in (b).



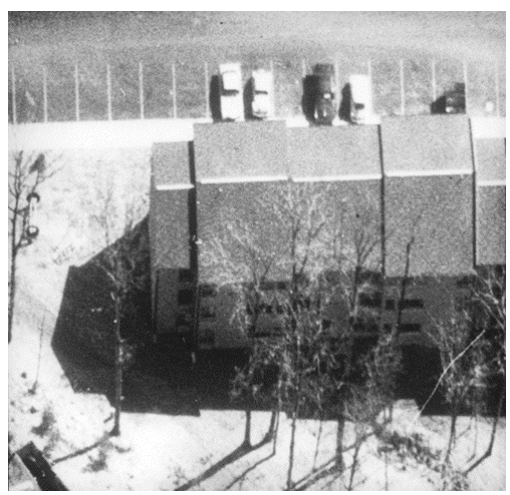
(a)



(b)

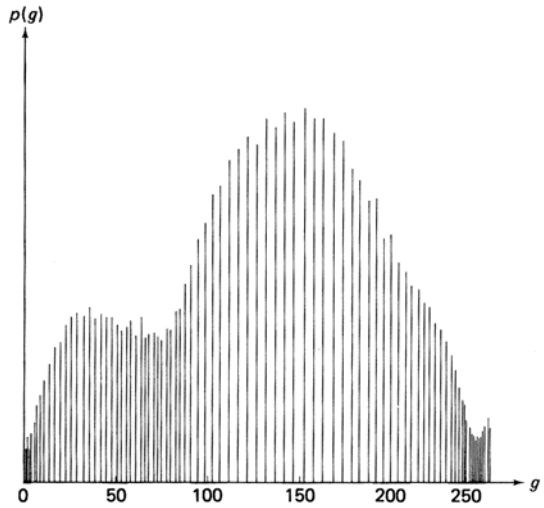


(c)



(d)

Figure 8.5 Example of gray scale modification. (a) Original image of 512 x 512 pixels; (b) histogram of the image in (a); (c) transformation function used in the gray scale modification; (d) processed image; (e) histogram of the processed image in (d).



(e)

Figure 8.5 (continued)

Histogram Matching

$T(r) = s$

Rather than $P_s(s)$ uniform, we want $P_s(s)$ to match a "desired" pdf given

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$r =$ pixel values before ~~[illegible]~~ matching

$z =$ pixel " after "

Can compute $P_r(r)$ from given image.

Know, given $P_z(z)$

<u>Goal</u>: what is the transform $r \longrightarrow z$?

Approach:

$$s = T(r) = \int_0^r p_r(w)\,dw$$

CDF of r.

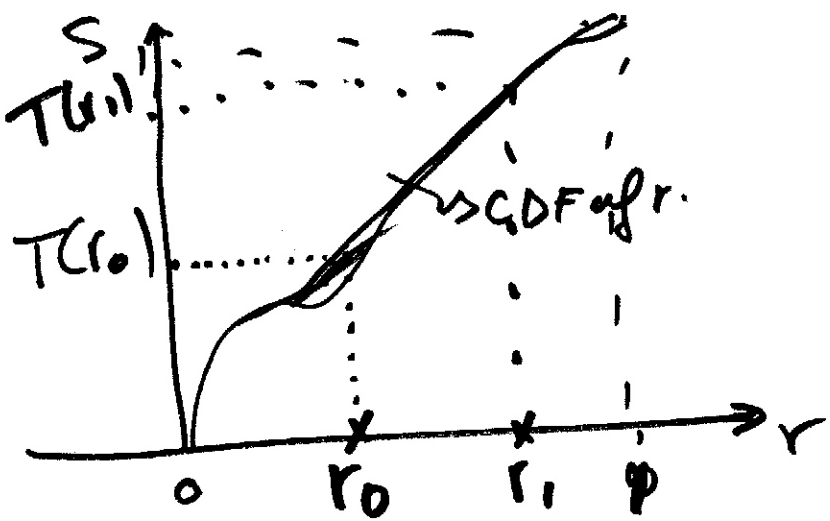
$$v = G(z) = \int_0^z p_z(t)\,dt$$

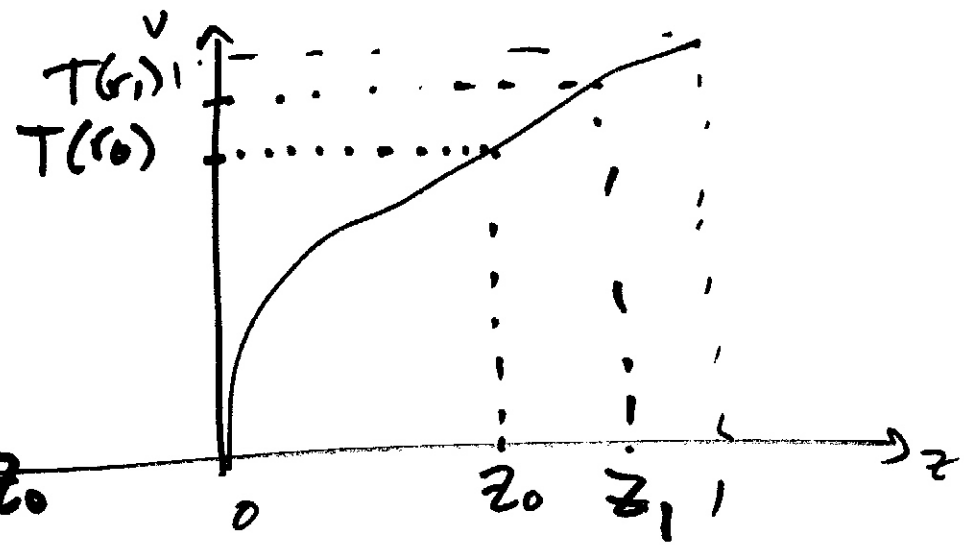
CDF of z.

Histogram matching

$G(z) = T(r)$

$z = G^{-1}\{T(r)\}$





$r_0 \longrightarrow$ how to find z_0

Lookup table

r_0	z_0
r_1	z_1

→ Matched histogram of $P_z(z)$ desired

Finding Transformation for Desired Histogram Modification

Input Histogram: $p(n_x)$

Desired Output Histogram: $P_d(n_y)$

Here we find a peak in the middle plot, typically smooth and spread out

From $p(n_x) \rightarrow p'(n_x)$: cumulative histogram

$$p'(n_x) = \sum_{k=0}^{n_x} p(k)$$

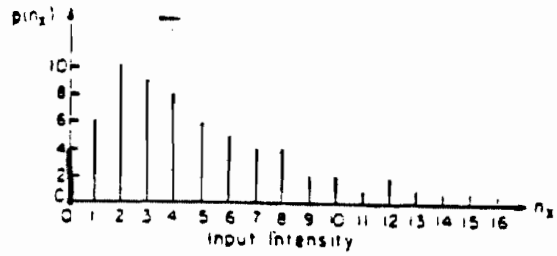
From $P_d(n_y) \rightarrow P'_d(n_y)$: desired cumulative histogram

$$P'_d(n_y) = \sum_{k=0}^{n_y} P_d(k)$$

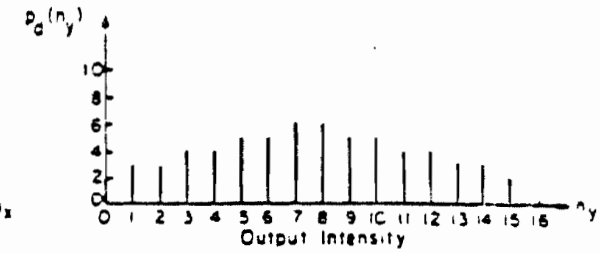
Choose transformation so that for each n_x , choose n_y so that $P_d'(n_y)$ is as close as possible to $p'(n_x)$

Example of Finding Transformation

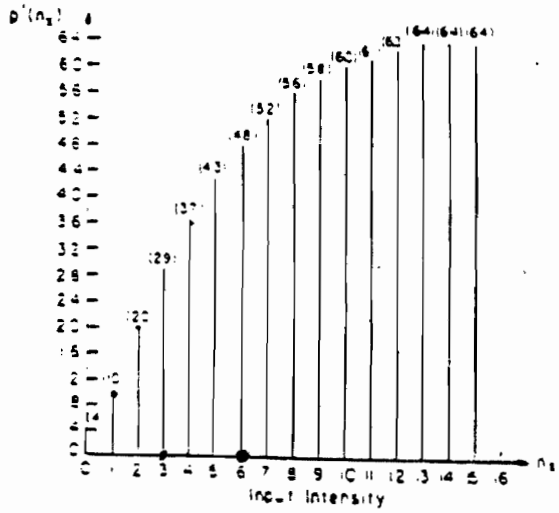
Input Histogram



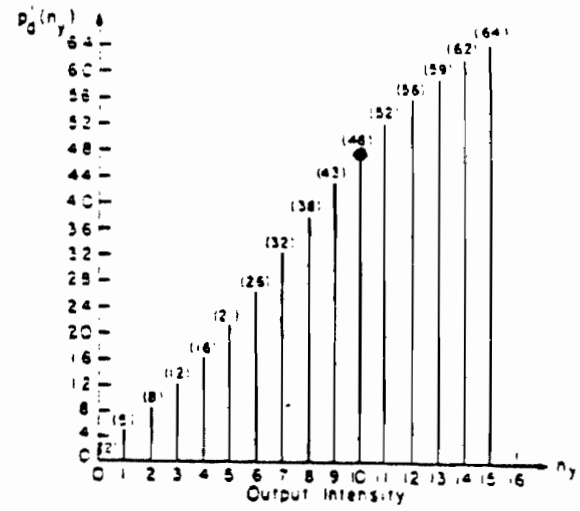
Desired Output Histogram



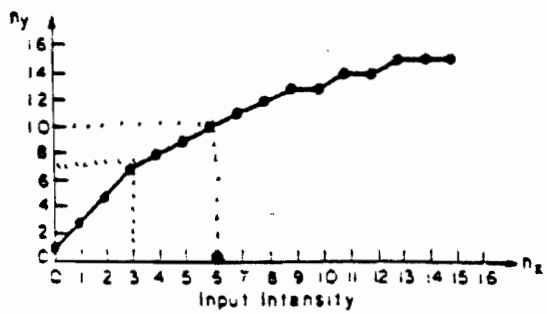
Cumulative Input Histogram



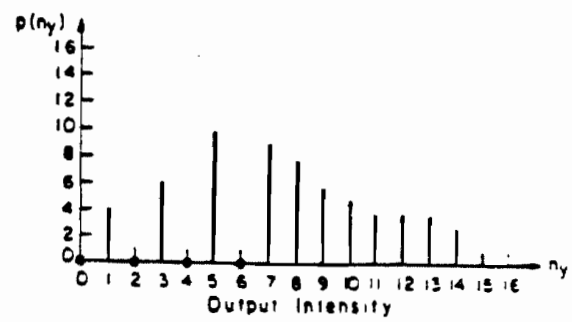
Cumulative Desired Output Histogram



Transformation Function



Output Histogram

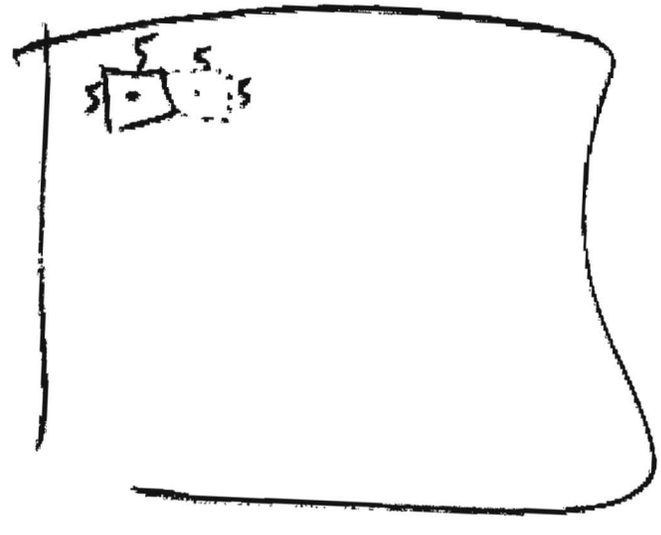


Local Enhancement
Using local histogram Analysis

Define square or rectangle neighborhood around each pixel, move center of "window" across the image.

At each location: compute histogram in the window

Do hist. eq.



Use local stats for Enhant

Stat $\rightarrow$ mean, variance:

global mean $= m_G = \sum_{i=0}^{L-1} r_i \, p(r_i)$ $\qquad r_i \rightarrow$ intensity level i

$i = 0, \ldots L-1$

global variance $= Var_G = \sum_{i=0}^{L-1} (r_i - m_G)^2 \, p(r_i)$

Local mean:

Let S_{xy} neighborhood, subimage centered at (x,y).

local mean $= m_{S_{xy}} = \sum_{(s,t) \in S_{xy}} r_{st} \, p(r_{st})$

local variance $= \sigma^2_{S_{xy}} = \sum_{(s,t) \in S_{xy}} [r_{st} - m_{S_{xy}}]^2 \, p(r_{st})$

Goal: Enhance dark areas while leaving the bright area as unchanged as possible.

- Detect region that have both.

(a) dark $\longrightarrow$ local mean has to be small compared to global mean.

$$m_{S_{xy}} \leq k_0 M_G$$

local mean

global mean

$0 \leq k_0 \leq 1$ constant

$k_0 \simeq 0.4$

(b) Low contrast

$$\sigma_{S_{xy}} \leq k_2 \, \mathrm{Var}_G$$

local var

global var

$k_2 \geq 0$ constant

$k_2 \simeq 0.4$

(c) not too low of contrast.
leave flat regions unchanged

$$\sigma_{S_{xy}} \geq K_1 \, Var_G$$

$K_1 \simeq 0.01$

$$g(x,y) = \text{processed} = \begin{cases} E\, f(x,y) & \text{if } m_{S_{xy}} \leq K_0 M_G \\ & K_1 \, Var_G \leq \sigma_{S_{xy}} \leq K_2 \, Var_G \\ 0 & \text{otherwise.} \end{cases}$$

Enhancement using Arithmetic / Logic Operations

Logic: AND . OR NOT → functionally complete.

NAND → functionally complete.

NOT → negative transformation

AND OR → Masking.

Image Subtraction

$$g(x,y) = f(x,y) - h(x,y)$$

mask.