Brief Detour into Random Processes

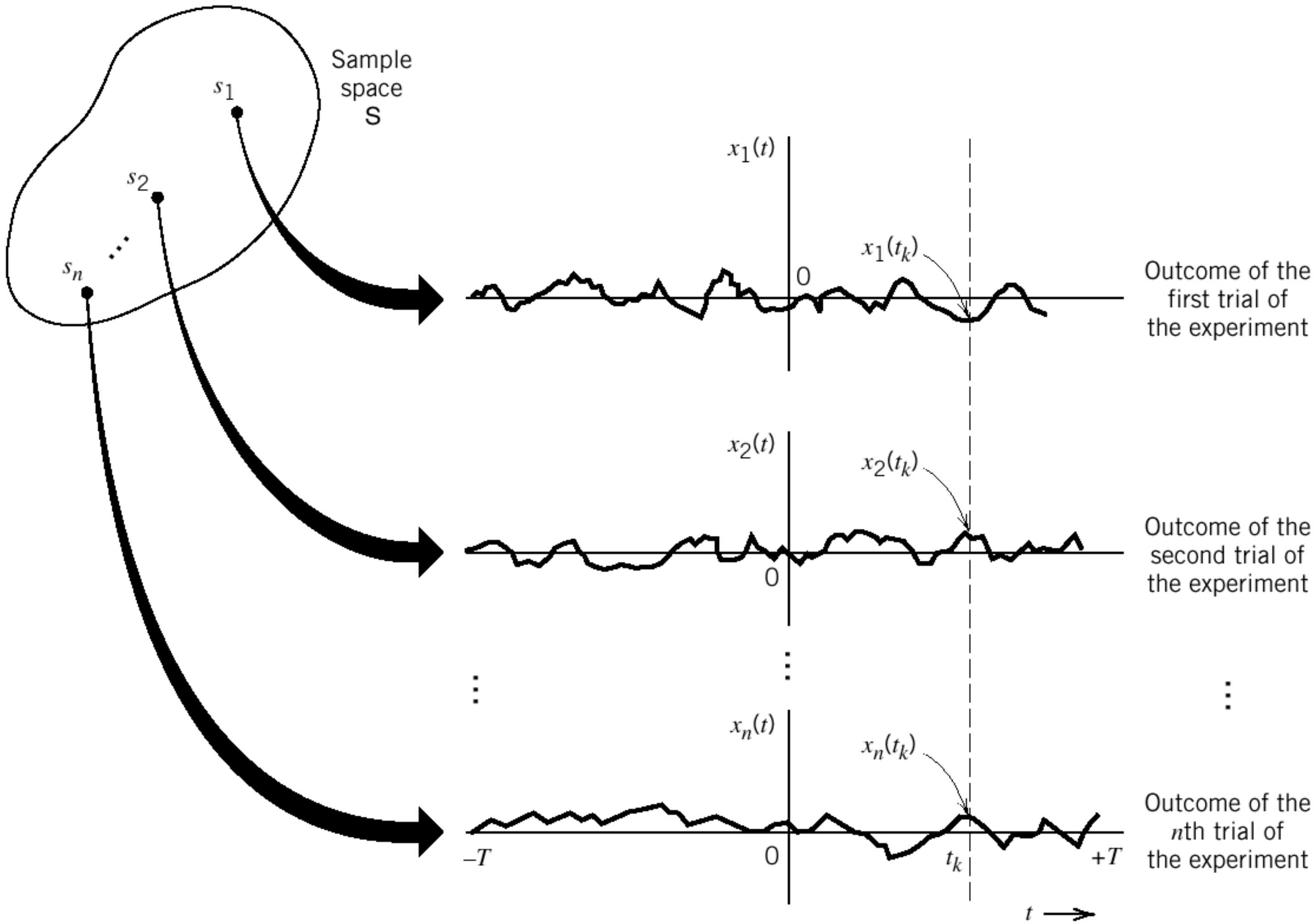


Figure 1.1 An ensemble of sample functions:

$$\{x_j(t) \mid j = 1,2,\ldots,n\}$$

$$\mathbf{S} \rightarrow X(t,s) \qquad -T \le t \le T \tag{1.1}$$

2T:The total observation interval

$$s_j \rightarrow X(t, s_j) = x_j(t) \tag{1.2}$$

$x_j(t)$= sample function

At $t = t_k$, $x_j(t_k)$ is a random variable (RV).

To simplify the notation , let $X(t,s) = X(t)$

$X(t)$:Random process, an ensemble of time function together with a probability rule.

Difference between RV and RP

RV: The outcome is mapped into a number

RP: The outcome is mapped into a function of time

1.3 Stationary Process

Stationary Process :

The statistical characterization of a process is independent of the time at which observation of the process is initiated.

Nonstationary Process:

Not a stationary process (unstable phenomenon)

Consider $X(t)$ which is initiated at t = $-\infty$,

$X(t_1), X(t_2) \ldots, X(t_k)$ denote the RV obtained at $t_1, t_2 \ldots, t_k$

For the RP to be stationary in the strict sense (strictly stationary)

The joint distribution function

$$F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k) = F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots x_k) \qquad (1.3)$$

For all time shift τ, all k, and all possible choice of $t_1, t_2 \ldots, t_k$

1.4 Mean, Correlation,and Covariance Function

Let $X(t)$ be a strictly stationary RP

The mean of X(t) is

$$\mu_X(t) = E[X(t)]$$

$$= \int_{-\infty}^{\infty} x f_{X(t)}(x)\,\mathrm{d}\,x \qquad (1.6)$$

$$= \mu_X \qquad \text{for all } t \qquad (1.7)$$

$f_{X(t)}(x)$: the first order pdf.

The autocorrelation function of $X(t)$ is

$$\mathrm{R}_X(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1,x_2)dx_1dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(0)X(t_2-t_1)}(x_1,x_2)dx_1dx_2$$

$$= R_X(t_2 - t_1) \qquad \text{for all } t_1 \text{ and } t_2 \qquad (1.8)$$

Properties of the autocorrelation function

For convenience of notation , we redefine

$$R_X(\tau) = E\left[X(t-\tau)X(t)\right], \qquad \text{for all } t \qquad (1.11)$$

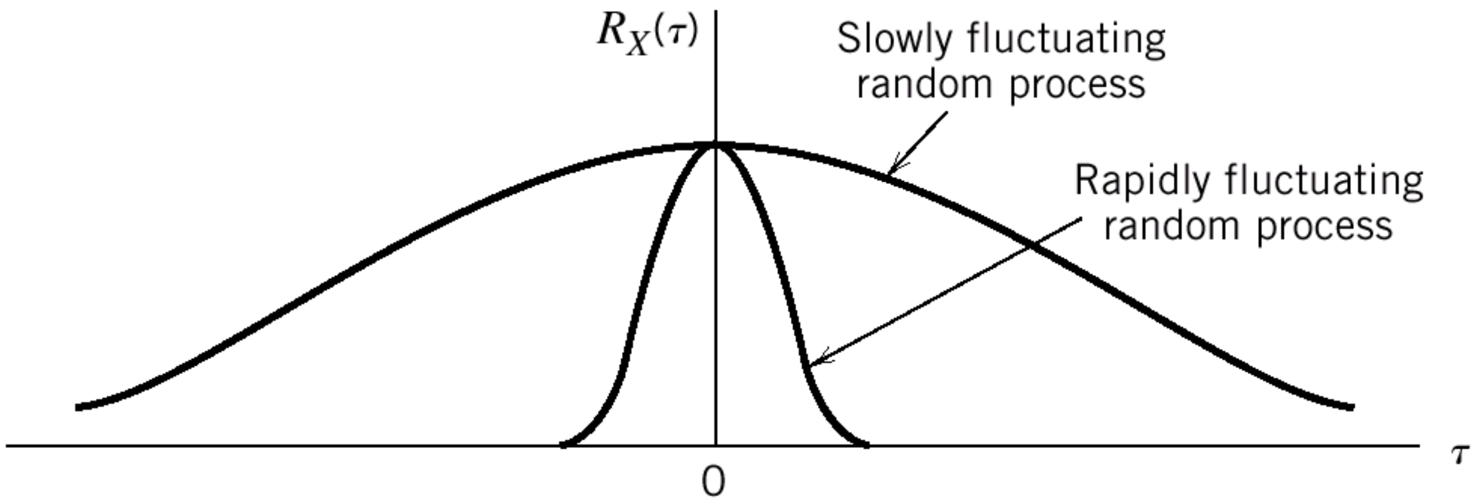
1. The mean-square value

$$R_X(0) = E\left[X^2(t)\right], \ \tau = 0 \qquad (1.12)$$

2. $R_X(\tau) = R(-\tau)$ (1.13)

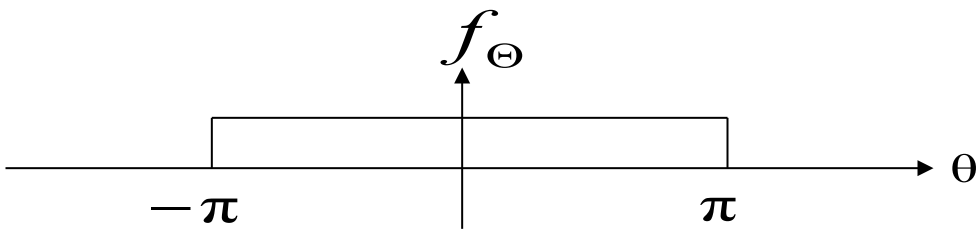
3. $\left|R_X(\tau)\right| \le R_X(0)$ (1.14)

The $R_X(\tau)$ provides the interdependence information of two random variables obtained from $X(t)$ at times τ seconds apart

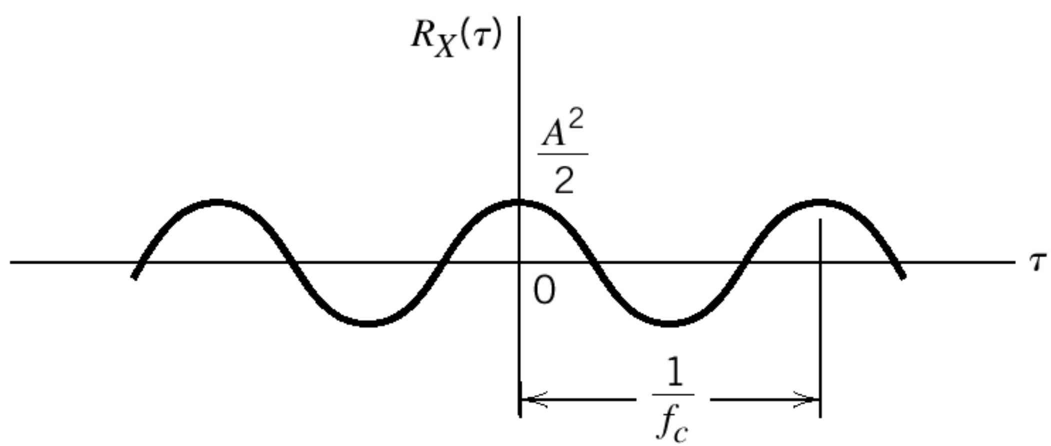


Example 1.2 $X(t) = A\cos(2\pi f_c t + \Theta)$ (1.15)

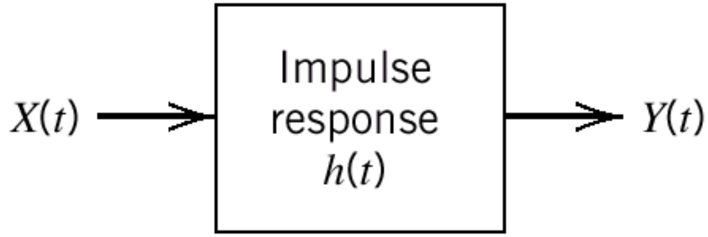
$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{elsewhere} \end{cases} \quad (1.16)$$



$$R_X(\tau) = E\left[X(t+\tau)X(t)\right] = \frac{A^2}{2}\cos(2\pi f_c t) \quad (1.17)$$



1.6 Transmission of a random Process Through a Linear Time-Invariant Filter (System)



$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1) X(t-\tau_1)\, d\tau_1$$

where h(t) is the impulse response of the system

$$\mu_Y(t) = E[Y(t)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1) X(t-\tau_1)\, d\tau_1\right] \qquad (1.27)$$

If $E[X(t)]$ is finite and system is stable

$$= \int_{-\infty}^{\infty} h(\tau_1) E[x(t-\tau_1)]\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1) \mu_X(t-\tau_1)\, d\tau_1 \qquad (1.28$$

If $X(t)$ is stationary, $\mu_Y = \mu_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 = \mu_X H(0)$, (1.29)

$H(0)$:System DC response.

Consider autocorrelation function of $Y(t)$:

$$R_Y(t,\mu) = E\left[Y(t)Y(\mu)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(\mu-\tau_2)\,d\tau_2\right] \qquad (1.30)$$

If $\mathrm{E}[X^2(t)]$ is finite and the system is stable,

$$R_Y(t,\mu) = \int_{-\infty}^{\infty} d\tau_1\, h(\tau_1) \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2) R_X(t-\tau_1, \mu-\tau_2) \qquad (1.31)$$

If $R_X(t-\tau_1, \mu-\tau_2) = R_X(t-\mu-\tau_1+\tau_2)$ (stationary)

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)\,d\tau_1\,d\tau_2 \qquad (1.32)$$

Stationary input, Stationary output

$$R_Y(0) = E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\,d\tau_1\,d\tau_2 \qquad (1.33)$$

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} df \left|H(f)\right|^2 \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau)\, d\tau \qquad (1.37)$$

$\left|H(f)\right|$: the magnitude response

Define: Power Spectral Density (Fourier Transform of R(τ))

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-2\pi f\tau)\, d\tau \qquad (1.38)$$

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} \left|H(f)\right|^2 S_X(f)\, df \qquad (1.39)$$

Recall $E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) R_X(\tau_2 - \tau_1)\, d\tau_1\, d\tau_2$ (1.33)

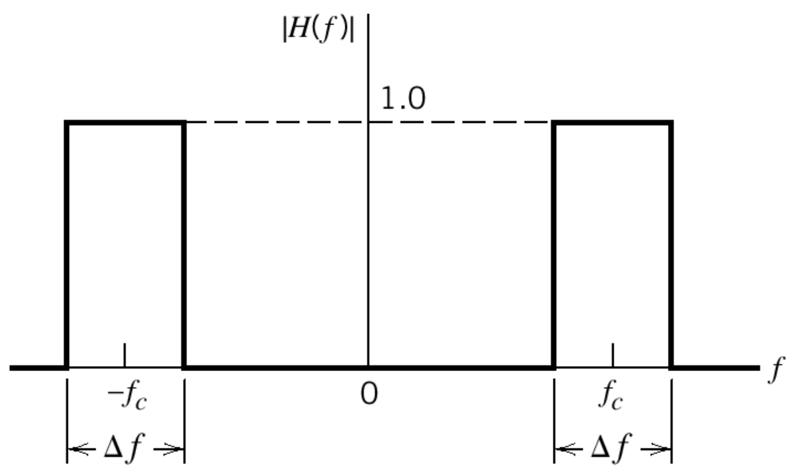
Let $\left|H(f)\right|$ be the magnitude response of an ideal narrowband filter

$$\left|H(f)\right| = \begin{cases} 1, & \left|f \pm f_c\right| < \frac{1}{2}\Delta f \\ 0, & \left|f \pm f_c\right| > \frac{1}{2}\Delta f \end{cases} \qquad (1.40)$$

Δf: Filter Bandwidth

If $\Delta f \ll f_c$ and $S_X(f)$ is continuous,

$E\left[Y^2(t)\right] \approx 2\Delta f\, S_X(f_c)$ in W/Hz



Properties of The PSD

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau)\, d\tau \qquad (1.42)$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(\tau) \exp(j2\pi f\tau)\, df \qquad (1.43)$$

Einstein-Wiener-Khintahine relations:

$$S_X(f) \Leftrightarrow R_X(\tau)$$

$S_X(f)$ is more useful than $R_X(\tau)$!

Example 1.5 Sinusoidal Wave with Random Phase

$$X(t) = A\cos(2\pi f_c t + \Theta),\ \Theta \sim U(-\pi, -\pi)$$

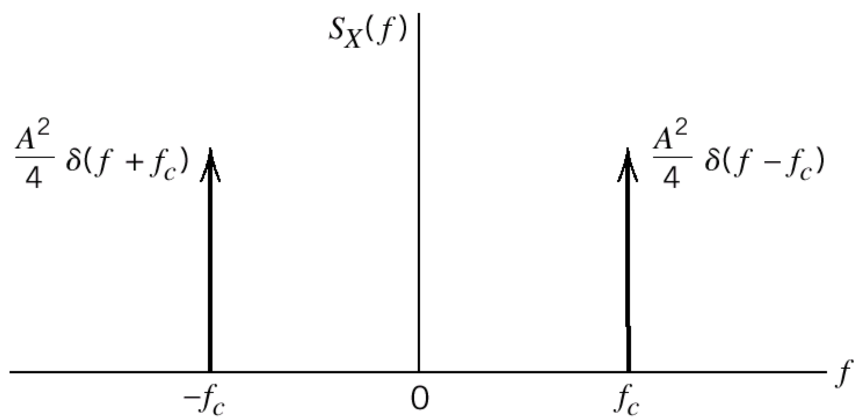
$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)\, d\tau$$

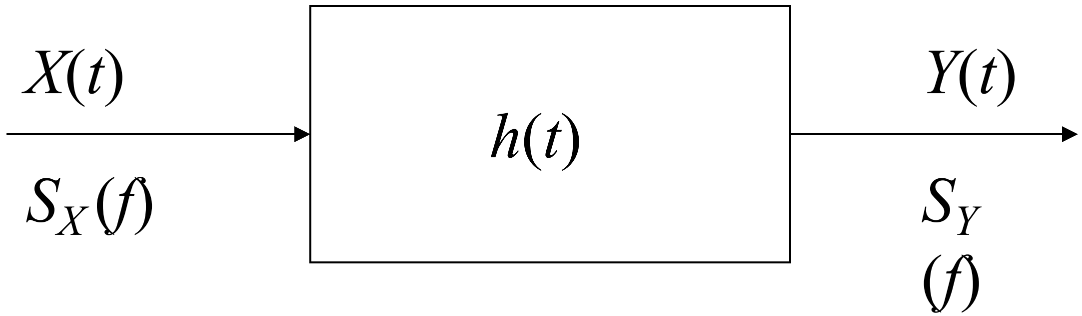
$$= \frac{A^2}{4}\int_{-\infty}^{\infty}\left[\exp(j2\pi f_c\tau)d\tau + \exp(-j2\pi f_c\tau)\right]\exp(-j2\pi f\tau)\, d\tau$$

$$= \frac{A^2}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

$\Rightarrow$ Appendix 2, $\int_{-\infty}^{\infty}\exp\left[j2\pi(f_c - f)\right]d\tau = \delta(f - f_c)$



Relation Among The PSD of The Input and Output Random Processes



Recall (1.32)

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)\, d\tau_1\, d\tau_2 \qquad (1.32)$$

$$S_Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2)\exp(-j2\pi f\tau)\, d\tau_1\, d\tau_2\, d\tau$$

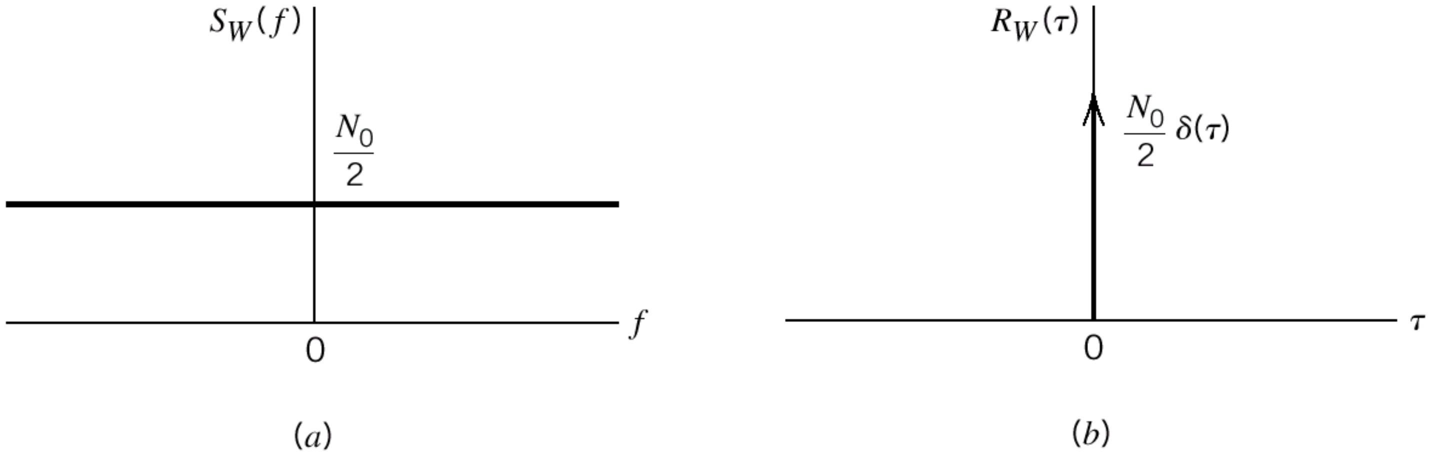
$Let\ \tau - \tau_1 + \tau_2 = \tau_0$, or $\tau = \tau_0 + \tau_1 - \tau_2$

$$S_Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_1)\exp(j2\pi f\tau_0)\exp(-j2\pi f\tau_2)\exp(-j2\pi f\tau_0)\, d\tau_1\, d\tau_2\, d\tau_0$$

$$= S_X(f)H(f)H^*(f)$$

$$= |H(f)|^2 S_X(f) \qquad (1.58)$$

· White noise



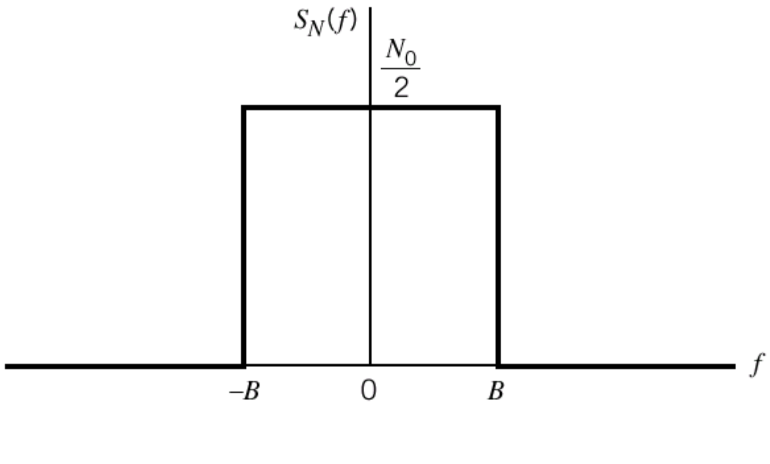
$$S_W(f) = \frac{N_0}{2} \qquad (1.93)$$

$$N_0 = kT_e \qquad (1.94)$$

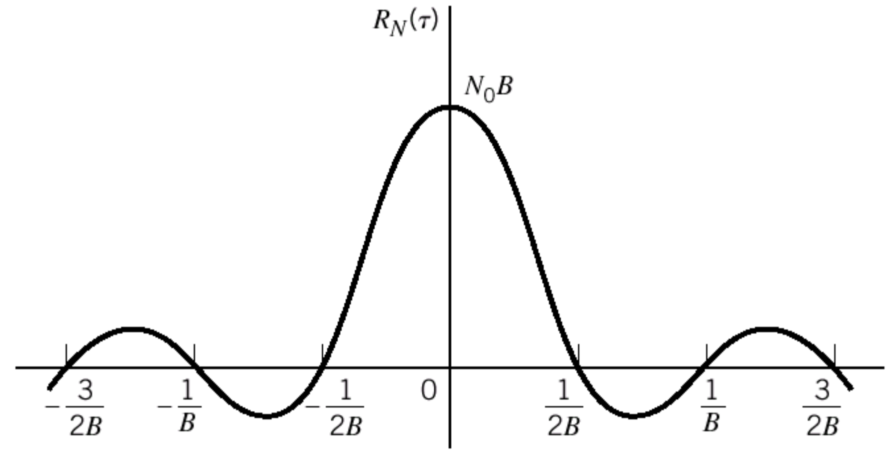
T_e : equivalent noise temperature of the receiver

$$R_W(\tau) = \frac{N_0}{2}\delta(\tau) \qquad (1.95)$$

Example 1.10 Ideal Low-Pass Filtered White Noise



(a)



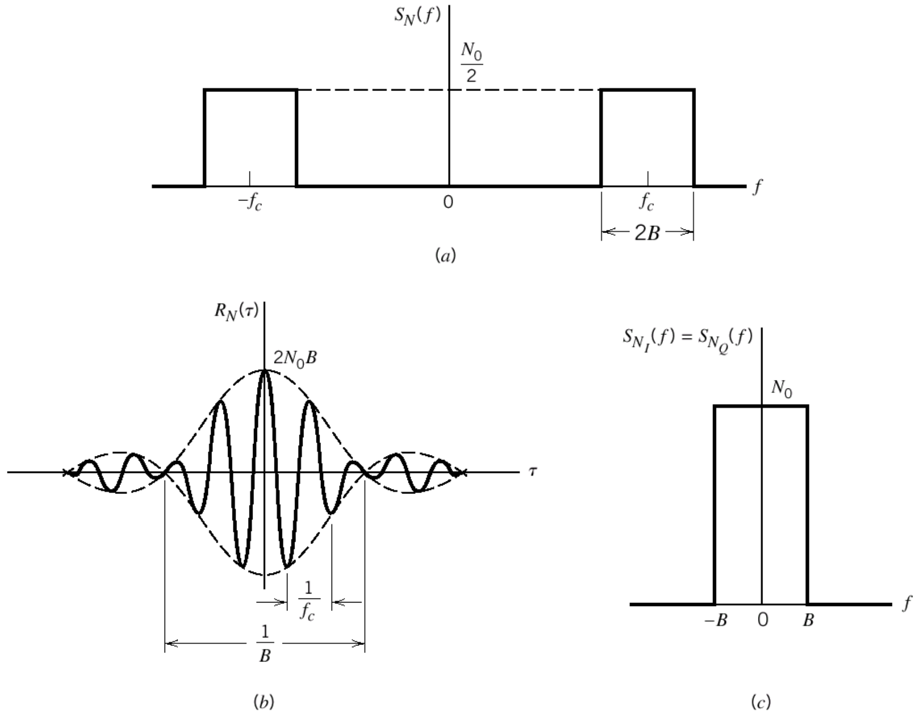
(b)

$$S_N(f) = \begin{cases} \dfrac{N_0}{2} & -B < f < B \\ 0 & |f| > B \end{cases} \quad (1.96)$$

$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f\tau)\, df \quad (1.97)$$

$$= N_0 B \operatorname{sinc}(2B\tau)$$

Example 1.12 Ideal Band-Pass Filtered White Noise



$$R_N(\tau) = \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau)df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau)df$$

$$= N_0 B \,\mathrm{sinc}(2B\tau)\left[\exp(-j2\pi f_c\tau)\exp(j2\pi f_c\tau)\right]$$

$$= 2N_0 B \,\mathrm{sinc}(2B\tau)\cos(2\pi f_c\tau) \qquad (1.103)$$

Compare with (1.97) (a factor of τ),

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = 2N_0 B \,\mathrm{sinc}(2B\tau).$$