FREQUENCY ANALYSIS

- Frequency Spectrum
  - Be basically the frequency components (spectral components) of that signal
  - Show what frequencies exists in the signal
- Fourier Transform (FT)
  - One way to find the frequency content
  - Tells how much of each frequency exists in a signal

$$X(k+1) = \sum_{n=0}^{N-1} x(n+1) \cdot W_N^{kn}$$

$$x(n+1) = \frac{1}{N} \sum_{k=0}^{N-1} X(k+1) \cdot W_N^{-kn}$$

$$w_N = e^{-j\left(\frac{2\pi}{N}\right)}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2j\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2j\pi ft} df$$

STATIONARITY OF SIGNAL

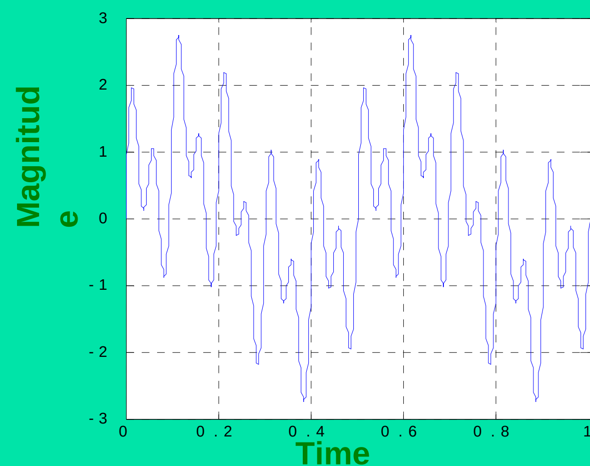
- Stationary Signal
  - Signals with frequency content unchanged in time
  - All frequency components exist at all times

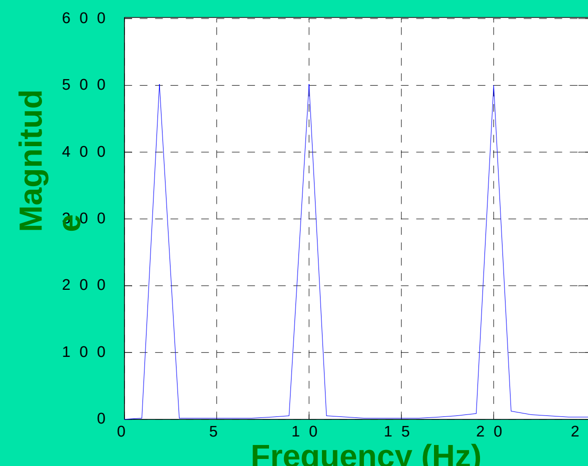
- Non-stationary Signal
  - Frequency changes in time
  - One example: the "Chirp Signal"

STATIONARITY OF SIGNAL

2 Hz + 10 Hz + 20Hz

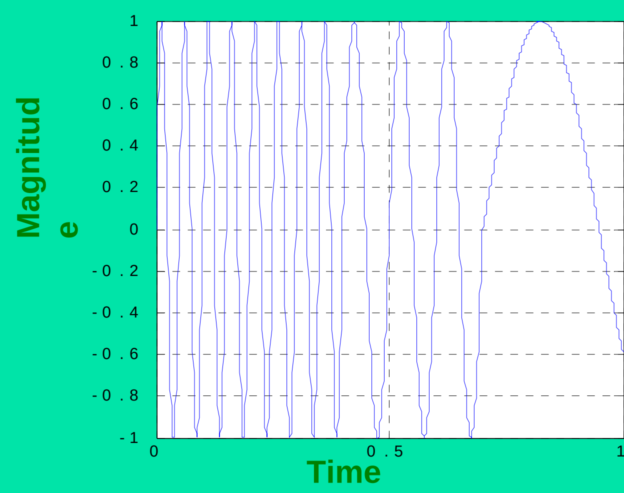
Stationary

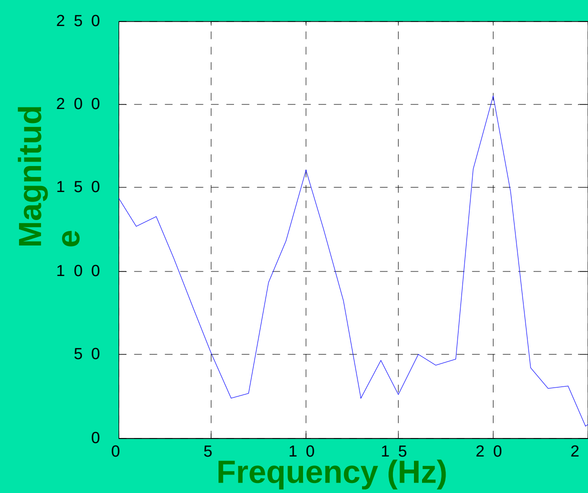




0.0-0.4: 2 Hz +
0.4-0.7: 10 Hz +
0.7-1.0: 20Hz

Non-Stationary



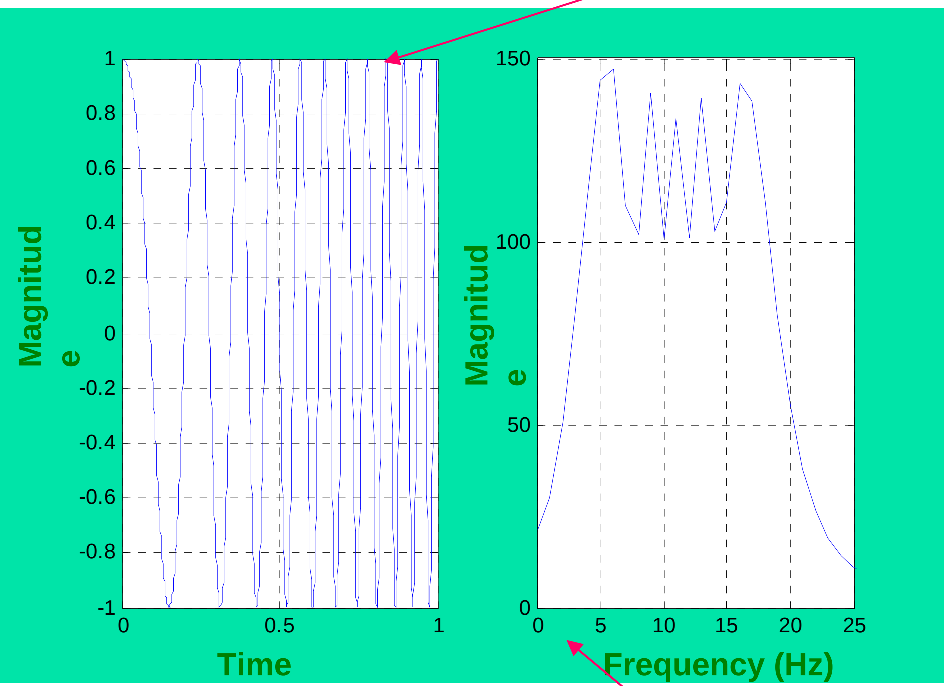


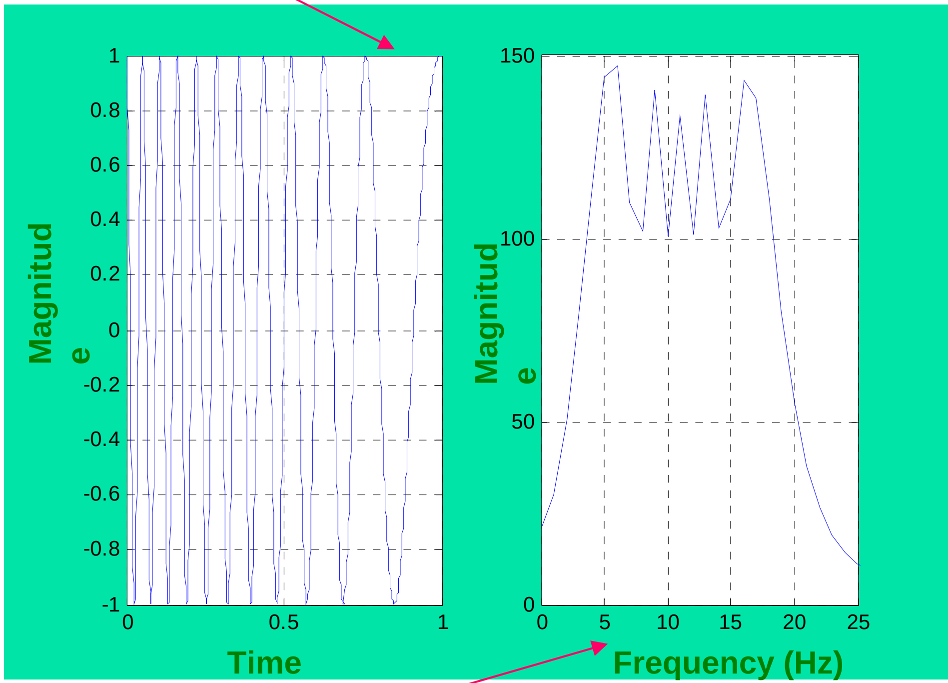
CHIRP SIGNALS

Frequency: 2 Hz to 20 Hz

Different in Time Domain

Frequency: 20 Hz to 2 Hz





Same in Frequency Domain

At what time the frequency components occur? FT can not tell!

NOTHING MORE, NOTHING LESS

- FT Only Gives what Frequency Components Exist in the Signal
- The Time and Frequency Information can not be Seen at the Same Time
- Time-frequency Representation of the Signal is Needed

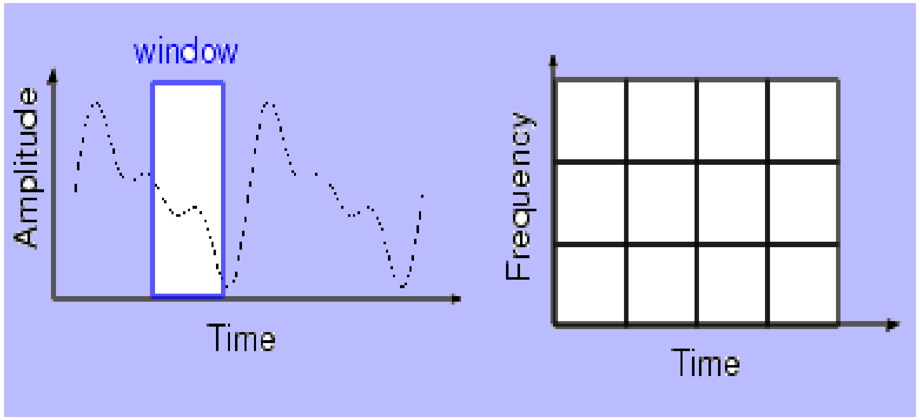
Most of Transportation Signals are Non-stationary.

(We need to know whether and also when an incident was happened.)

ONE EARLIER SOLUTION: SHORT-TIME FOURIER TRANSFORM (STFT)

SFORT TIME FOURIER TRANSFORM (STFT)

- Dennis Gabor (1946) Used STFT
  - To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.
- The Segment of Signal is Assumed *Stationary*
- A 3D transform



$$\mathrm{STFT}_X^{(\omega)}(t', f) = \int_t \left[x(t) \bullet \omega^*(t - t')\right] \bullet e^{-j2\pi ft}\, dt$$

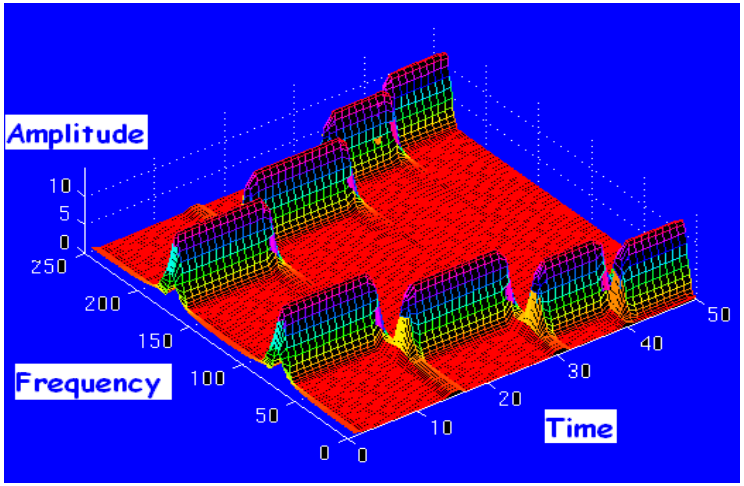
$\omega(t)$: the window function

A function of time and frequency

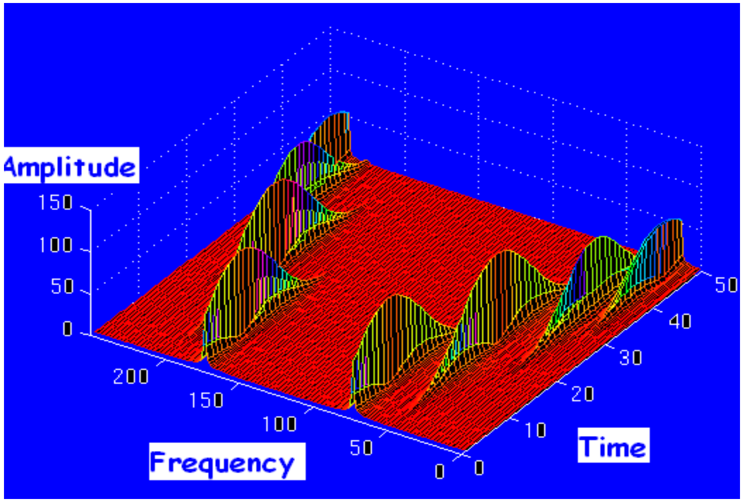
DRAWBACKS OF STFT

- Unchanged Window
- Dilemma of Resolution
  - Narrow window -> poor frequency resolution
  - Wide window -> poor time resolution
- Heisenberg Uncertainty Principle
  - Cannot know what frequency exists at what time intervals

Via Narrow Window

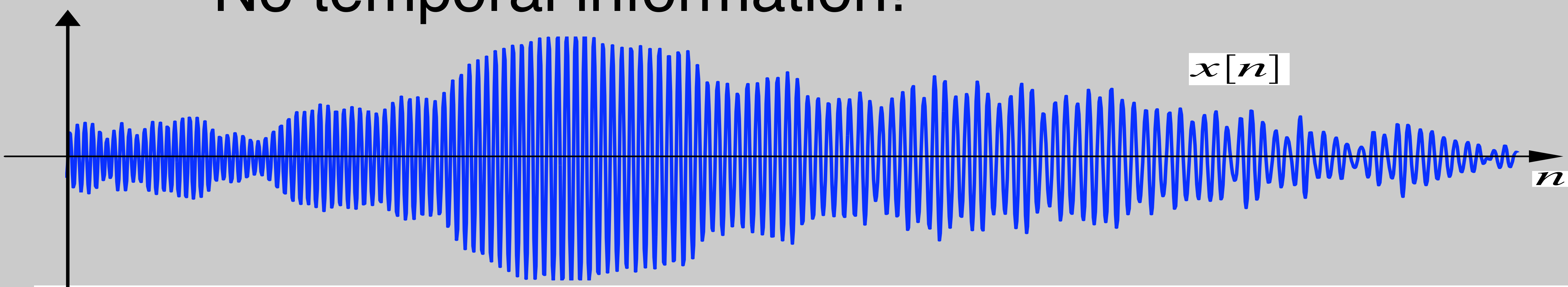


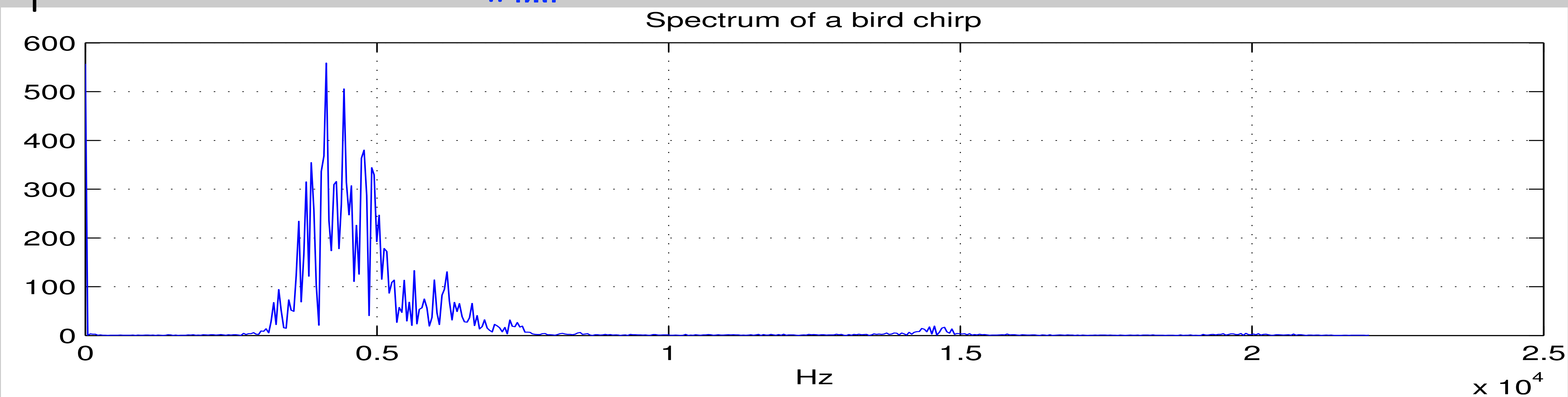
Via Wide Window



Example of spectral analysis

- Spectrum of a bird chirping
  - Interesting,.... but...
  - Does not tell the whole story
  - No temporal information!





Time Dependent Fourier Transform

- To get temporal information, use part of the signal around every time point

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$

*Also called Short-time Fourier Transform (STFT)

- Mapping from 1D ⇒ 2D, n discrete, *w* cont.

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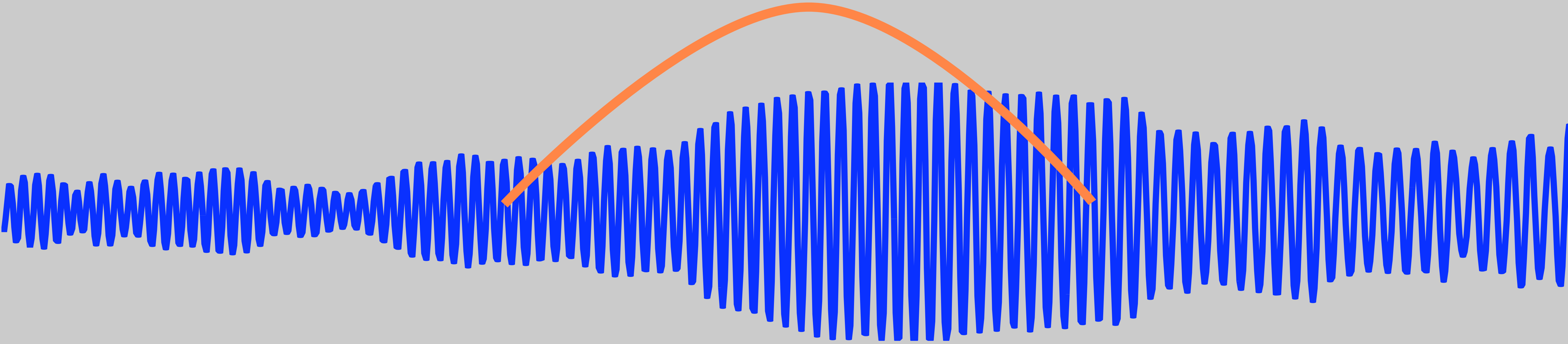
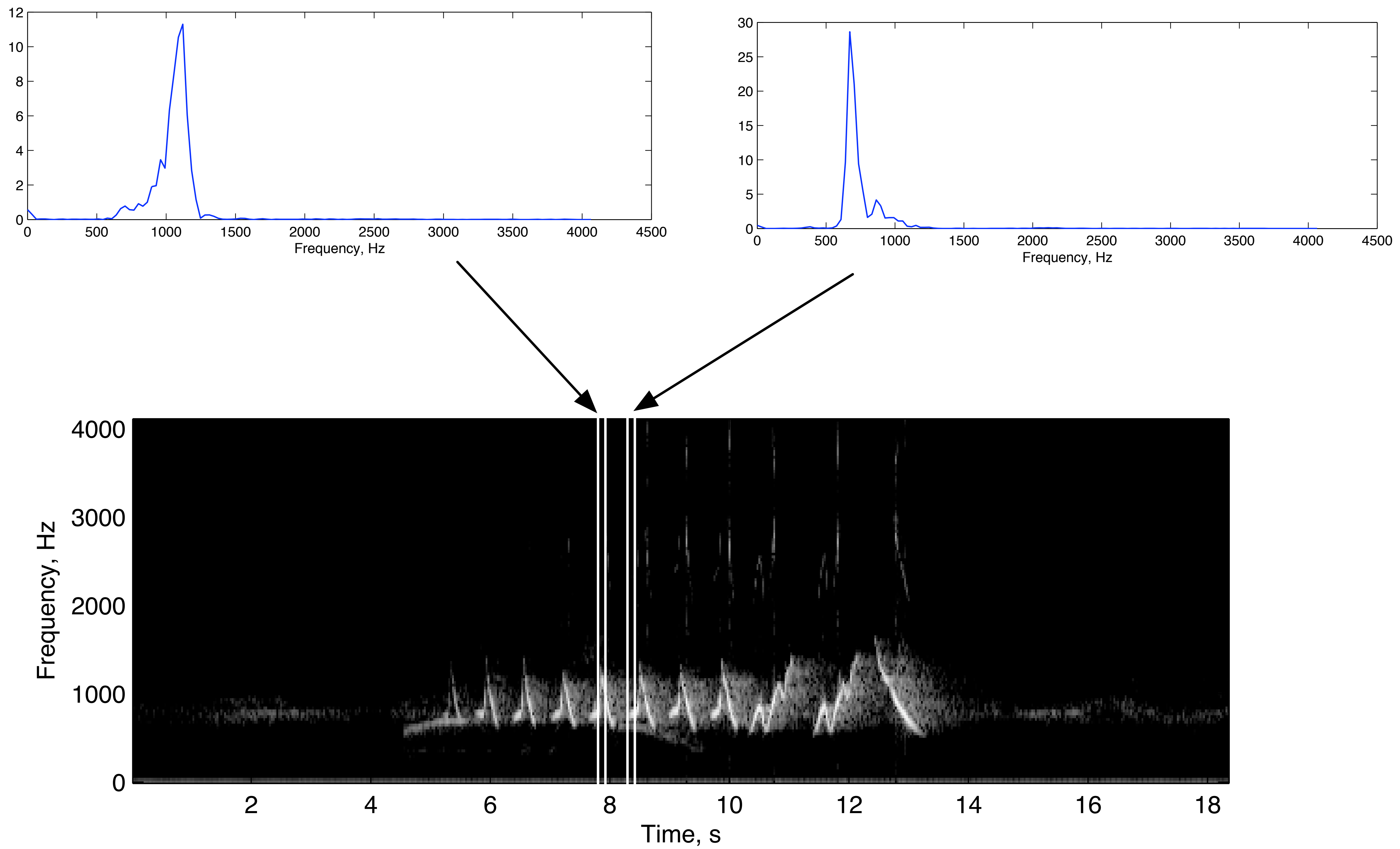


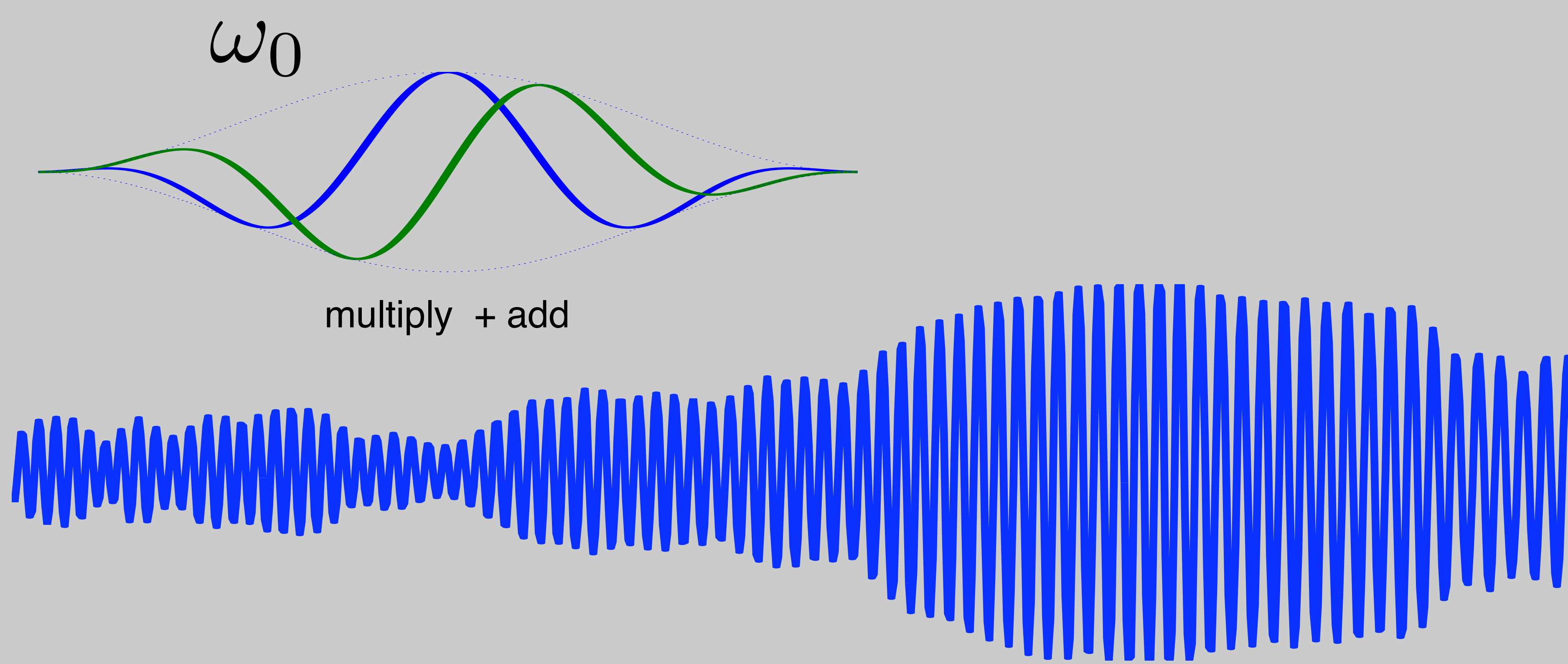
Spectrogram



Another view of STFT

- Can be expressed as a convolution

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$



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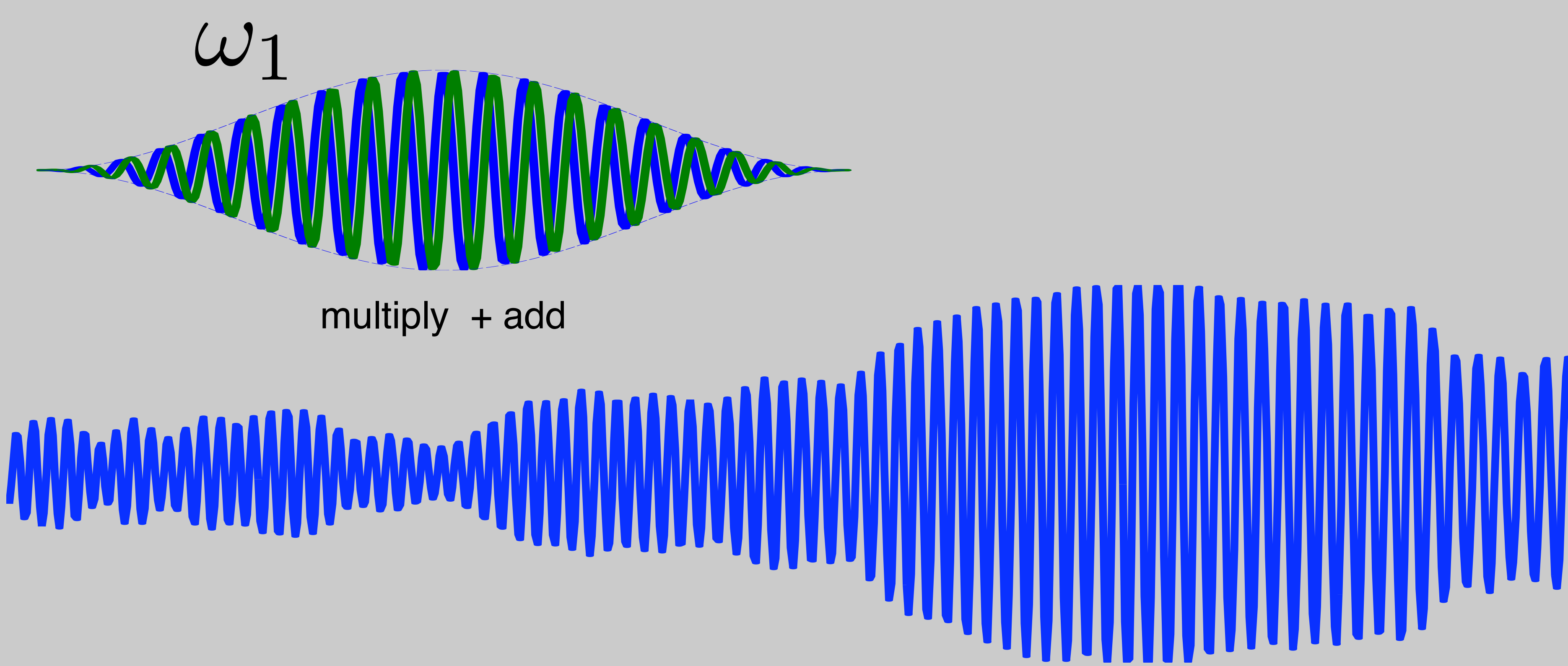
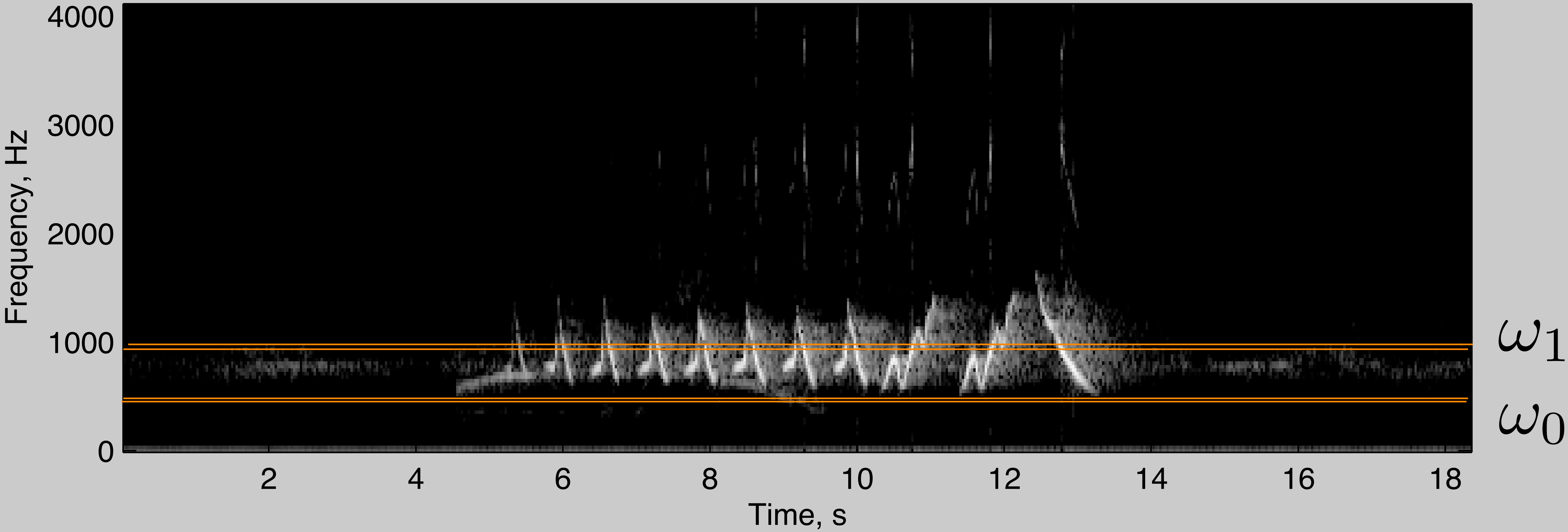
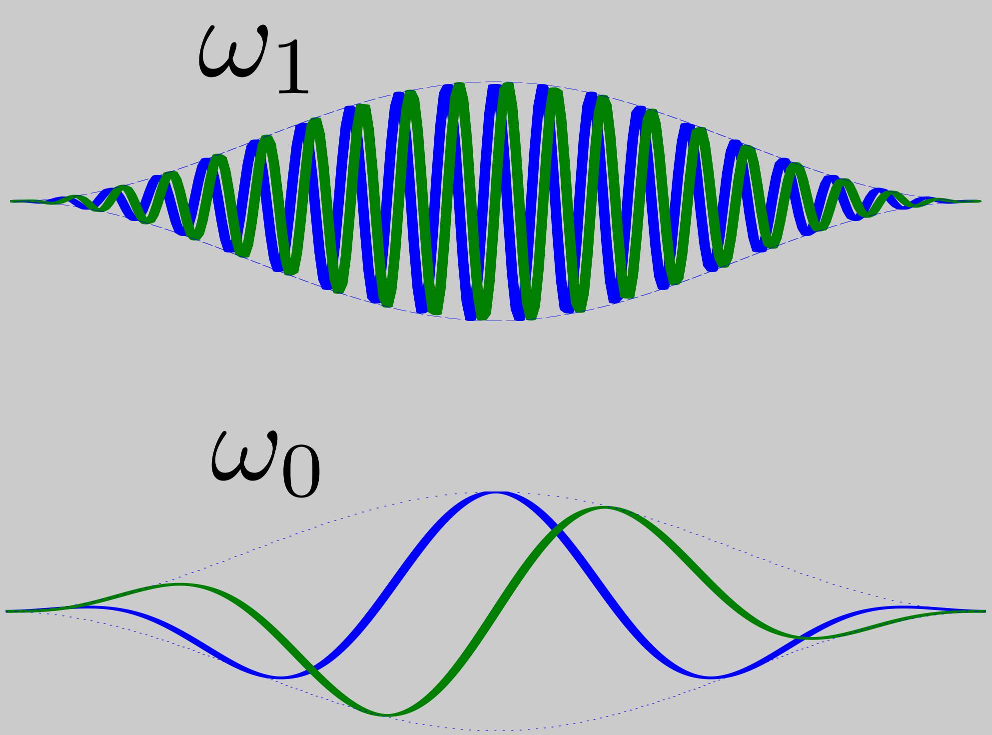
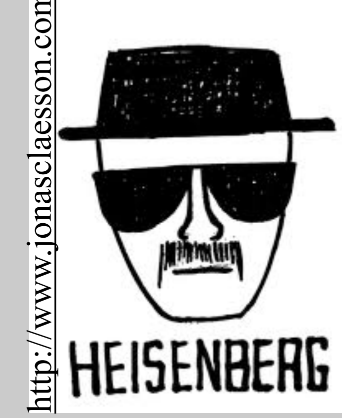


Basis functions (Atoms)



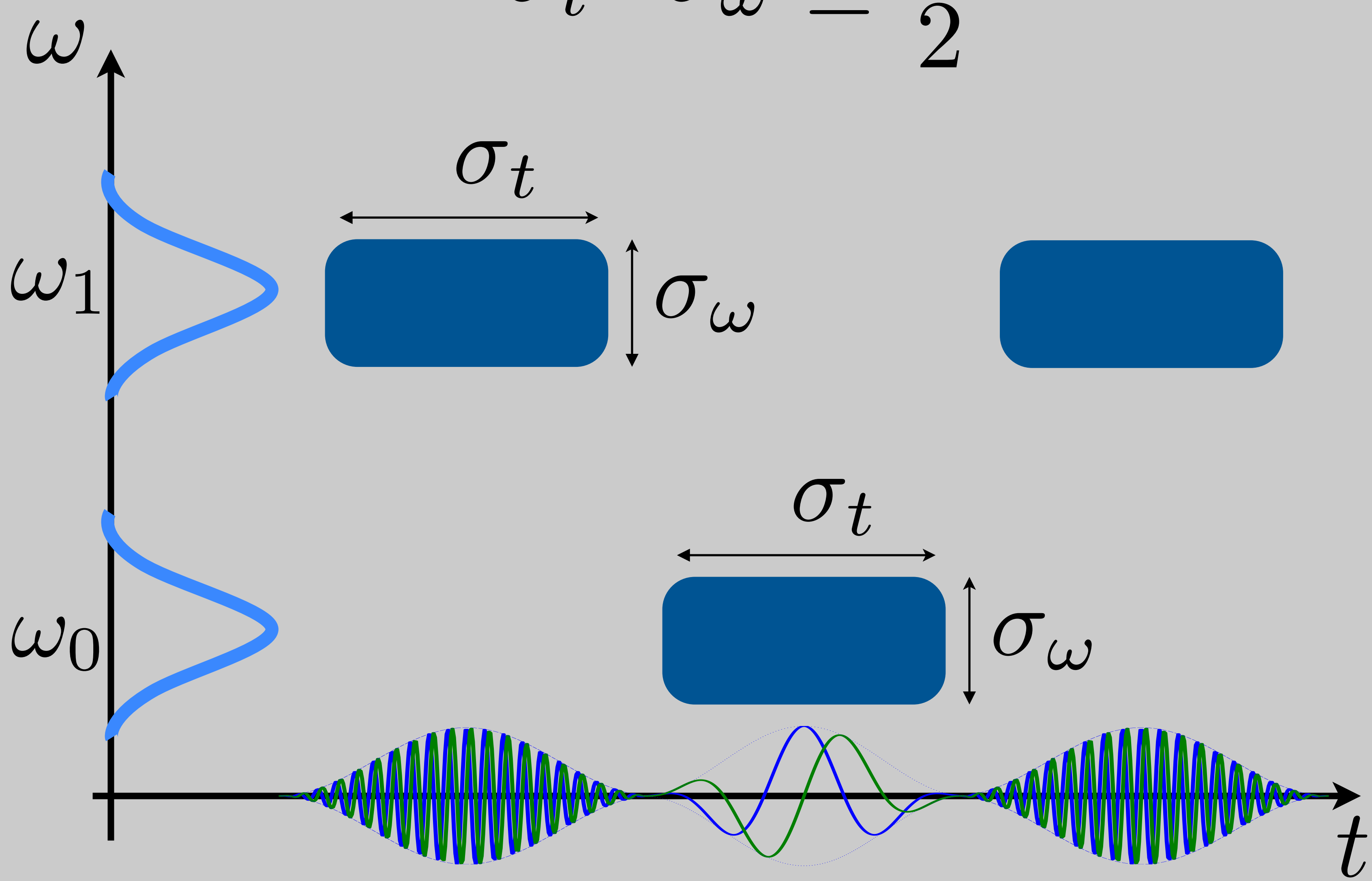


Heisenberg Boxes



- Time-Frequency uncertainty principle

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$



Window Characte[...] determi[...] σ_t an[...] σ_ω

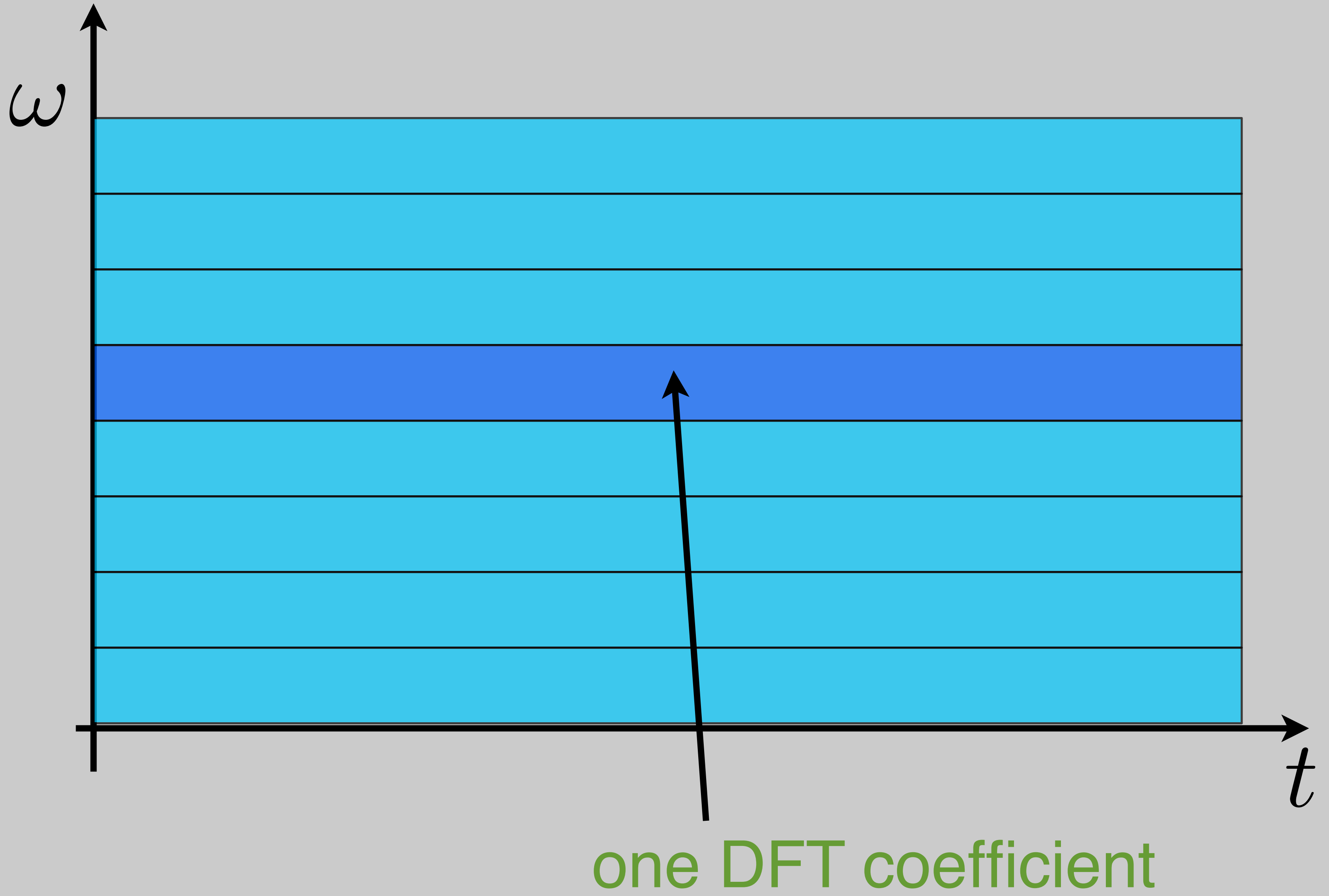
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$



Discrete Time Dependent FT

$$X_r[k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j2\pi km/N}$$

- L - Window length
- R - Jump of samples
- N - DFT length

- Tradeoff between time and frequency resolution

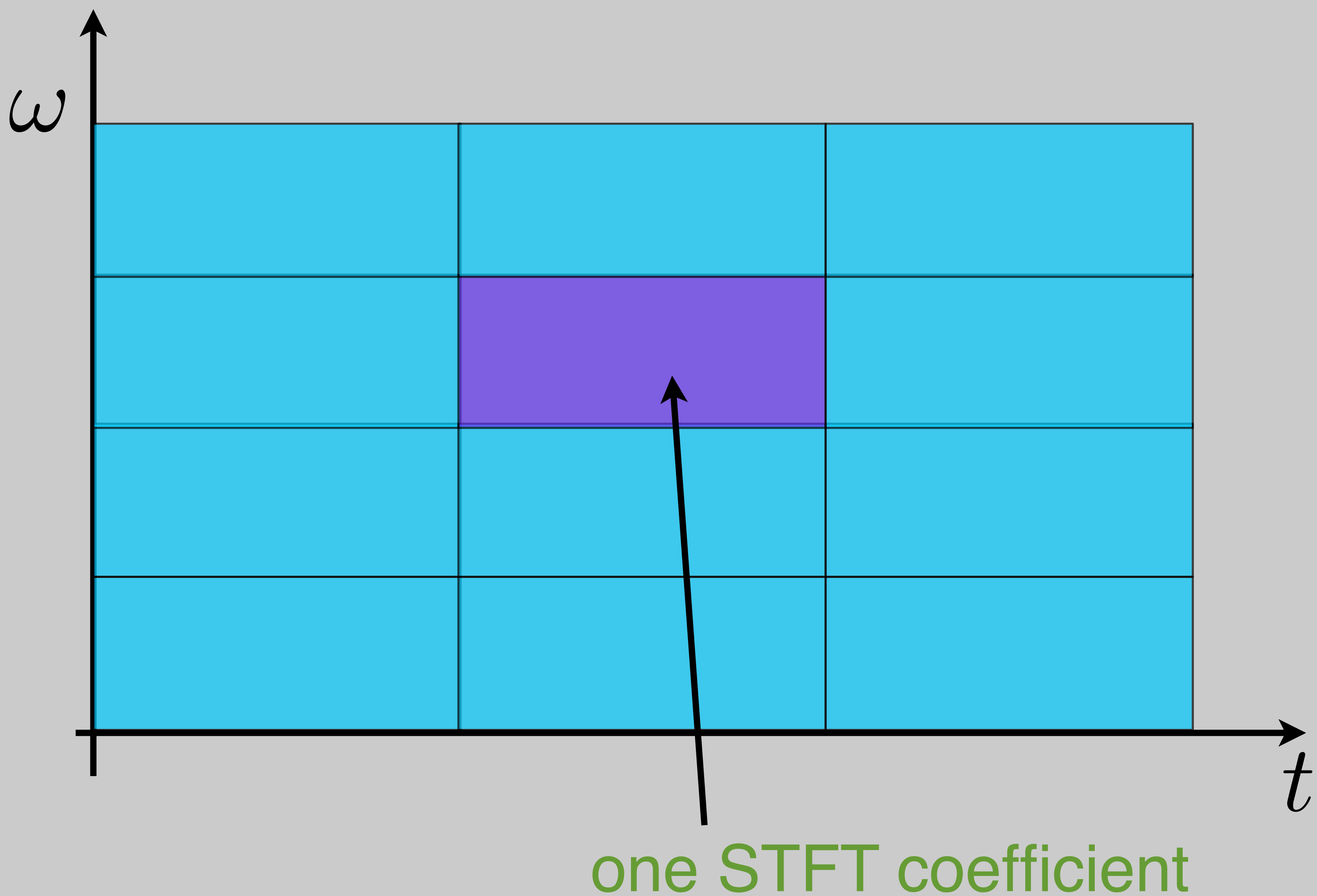
Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[r{\color{green}R} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



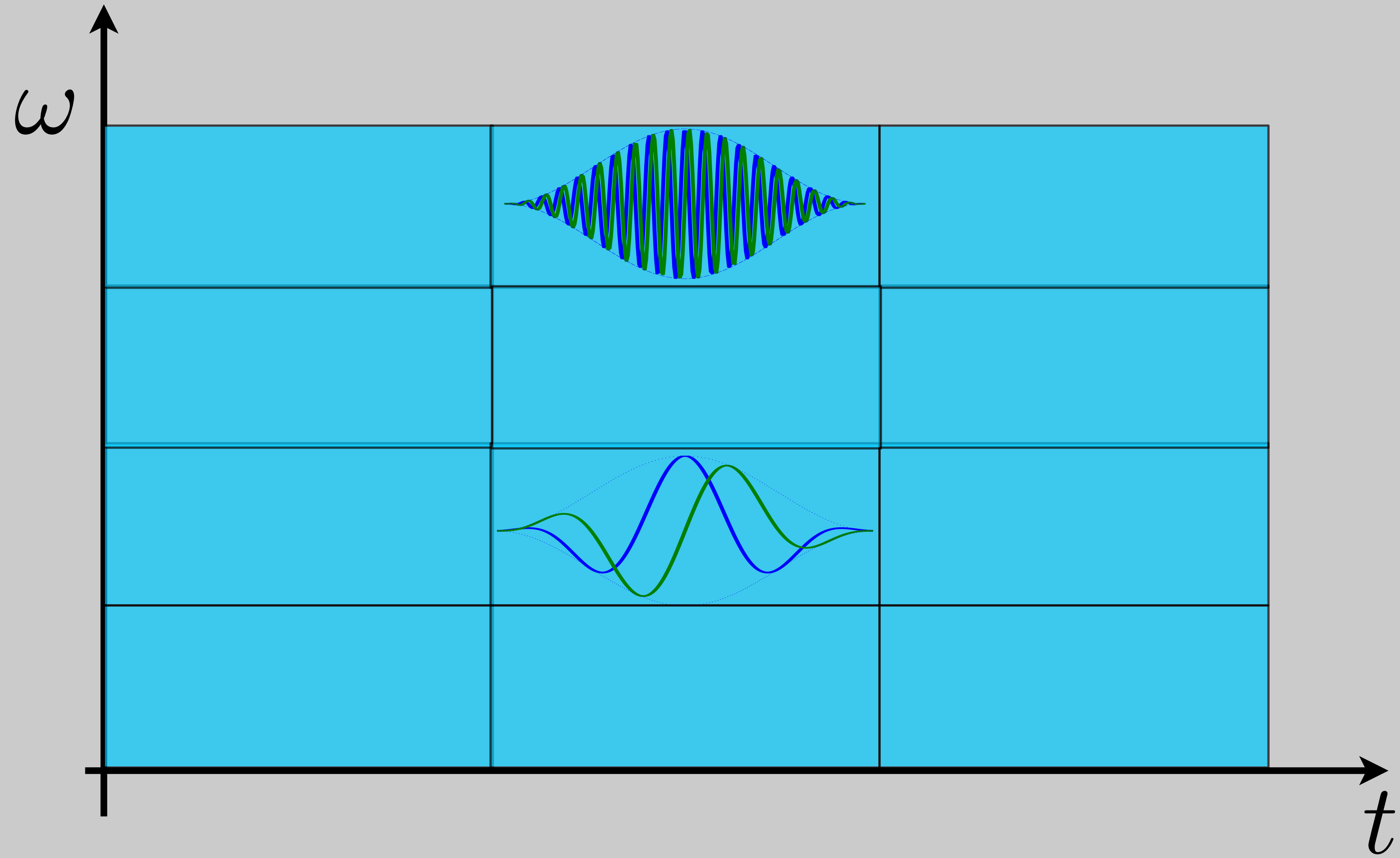
Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[rR + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$

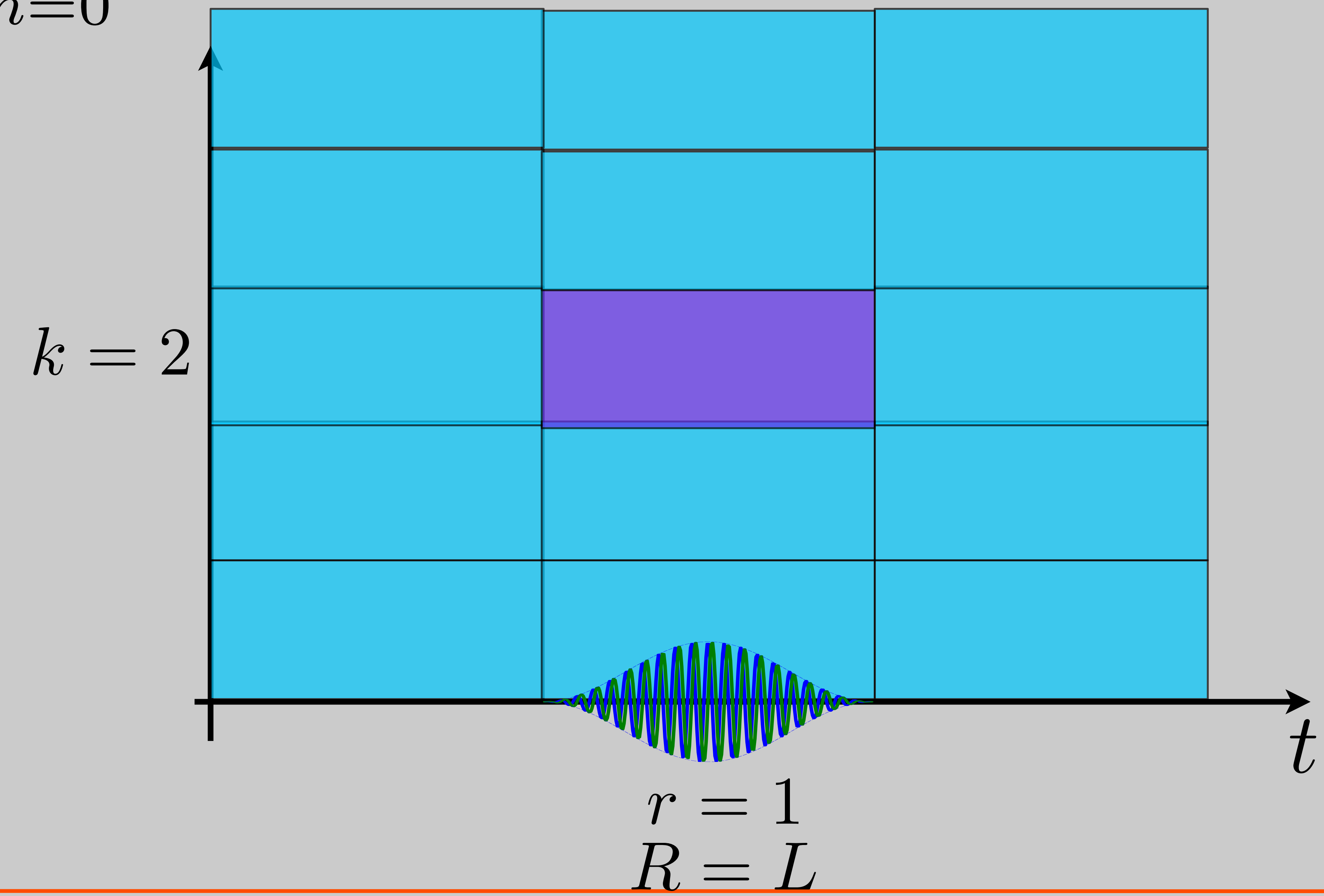


one STFT coefficient

Discrete STFT

optional ↓

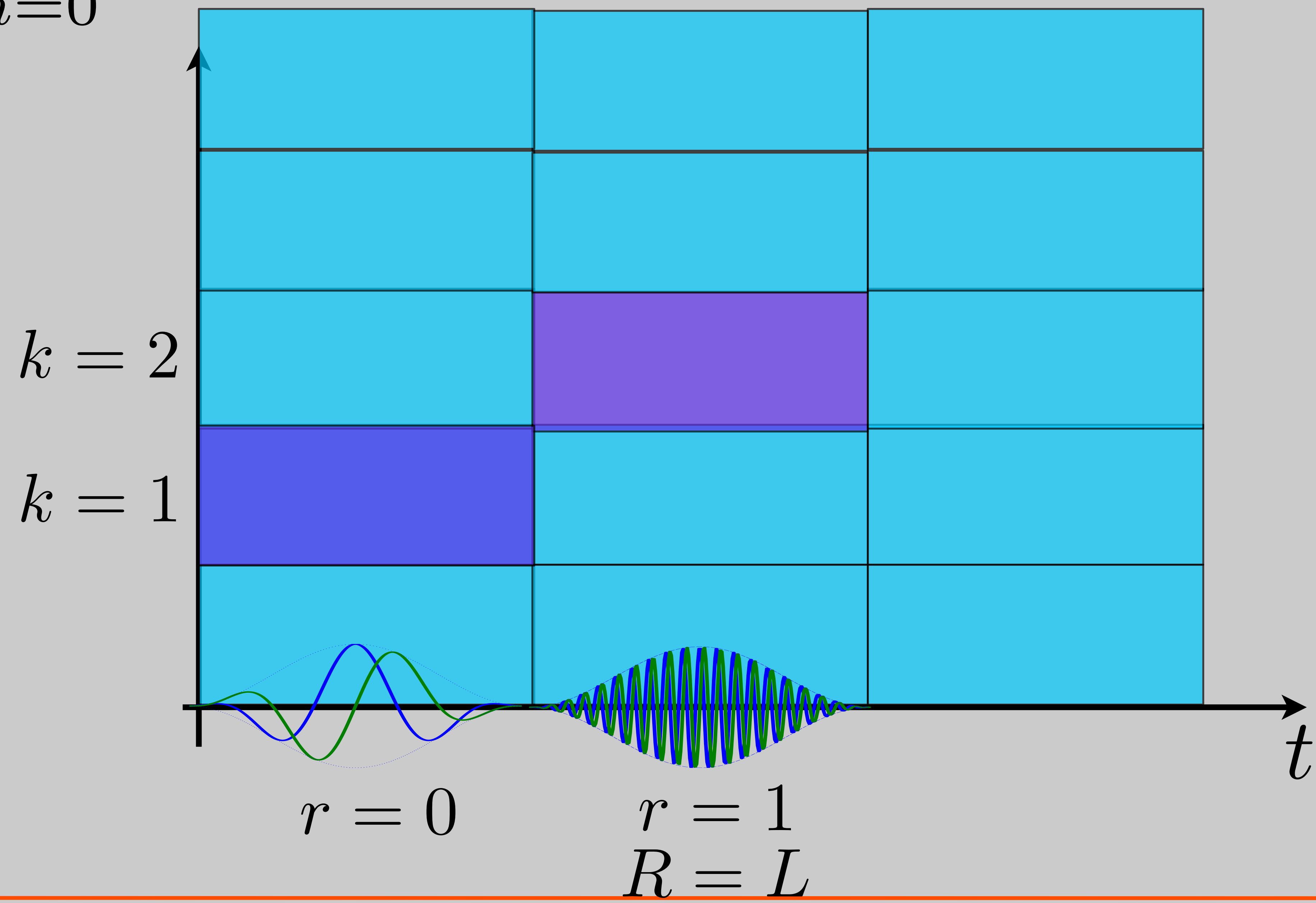
$$X[r,k] = \sum_{m=0}^{L-1} x[rR + m]w[m]e^{-j2\pi km/N}$$



Discrete STFT

optional ↓

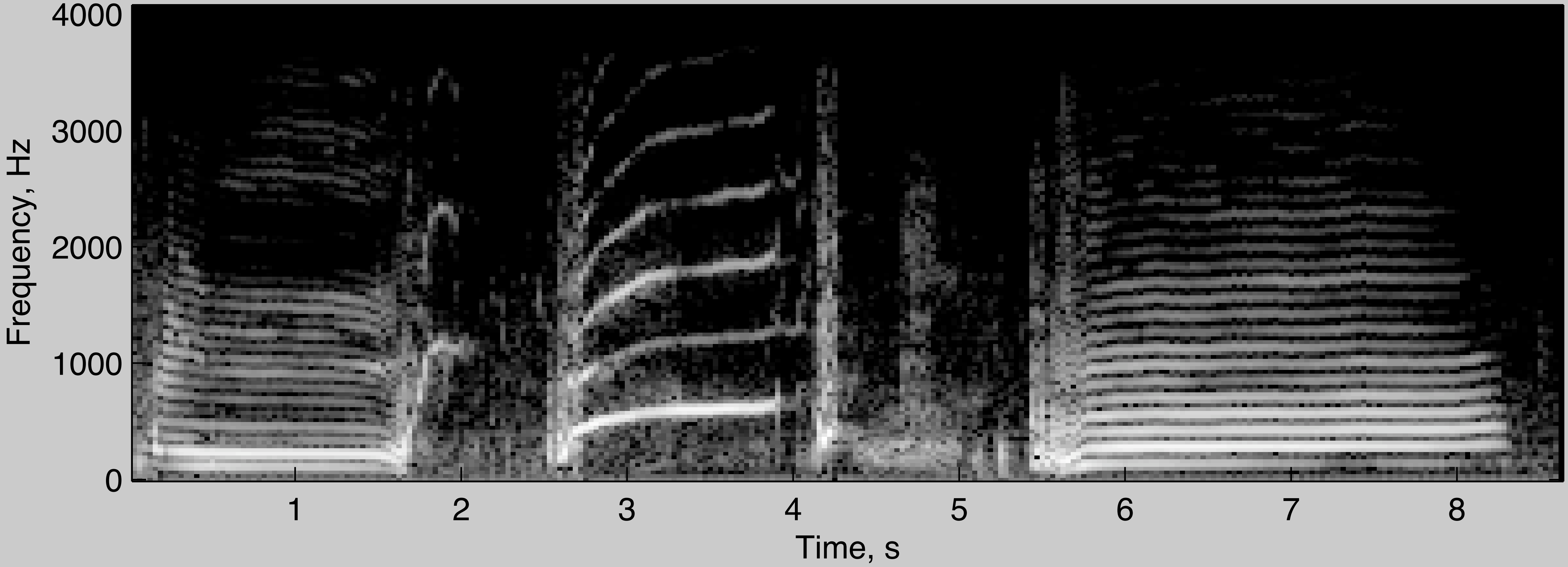
$$X[r,k] = \sum_{m=0}^{L-1} x[rR + m]w[m]e^{-j2\pi km/N}$$



Applications

- Time Frequency Analysis

Spectrogram of Orca whale

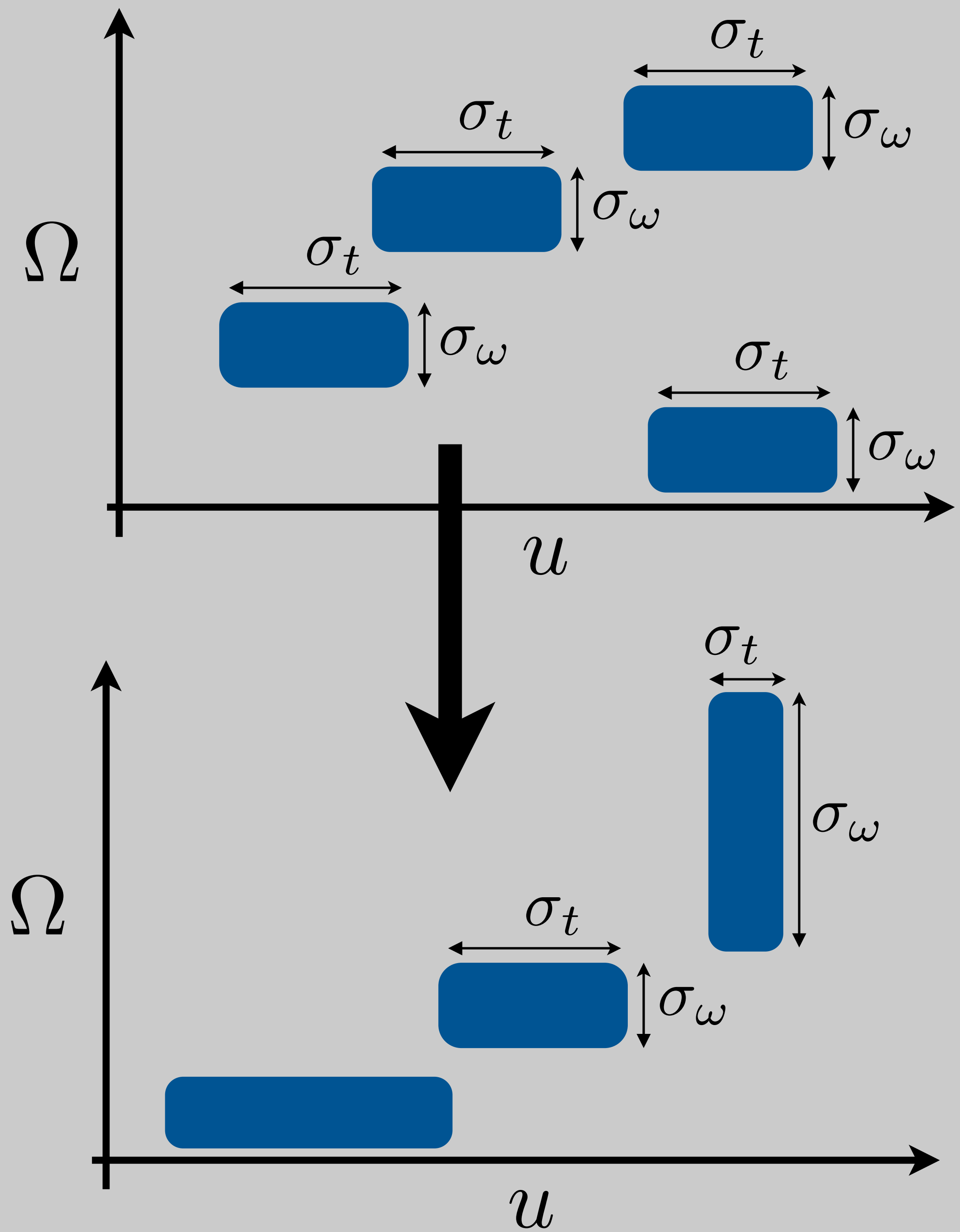


- Basic Idea:
  - low-freq changes slowly - fast tracking unimportant
  - Fast tracking of high-freq is important in many apps.
  - Must adapt Heisenberg box to frequency

- Back to continuous time for a bit.....

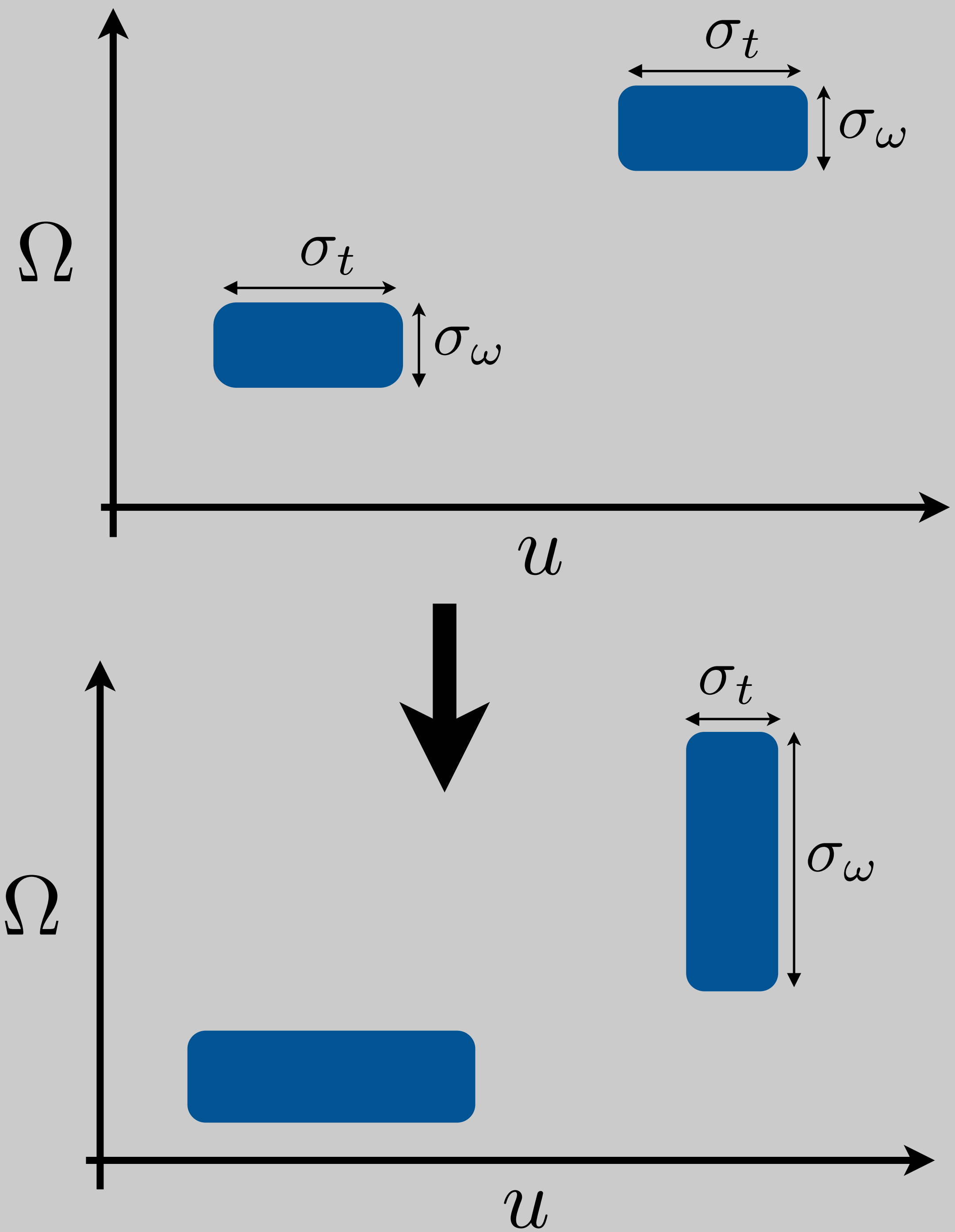
From STFT to Wavelets

- Continuous time



From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt$$

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*(\frac{t-u}{s}) dt$$

- The function Ψ is called a mother wavelet

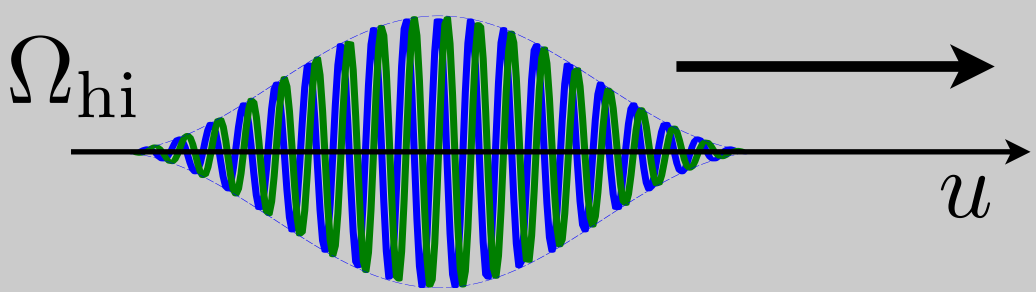
$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1$$ ⇒ unit norm

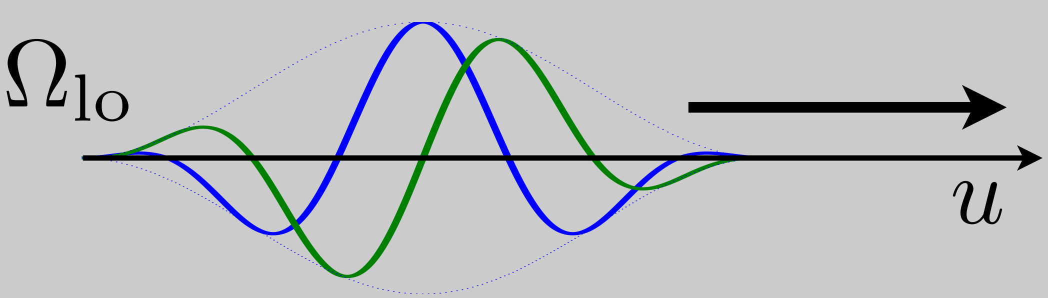
$$\int_{-\infty}^{\infty} \Psi(t) dt = 0$$ ⇒ Band-Pass

STFT and Wavelets "Atoms"

STFT Atoms
(with hamming window)

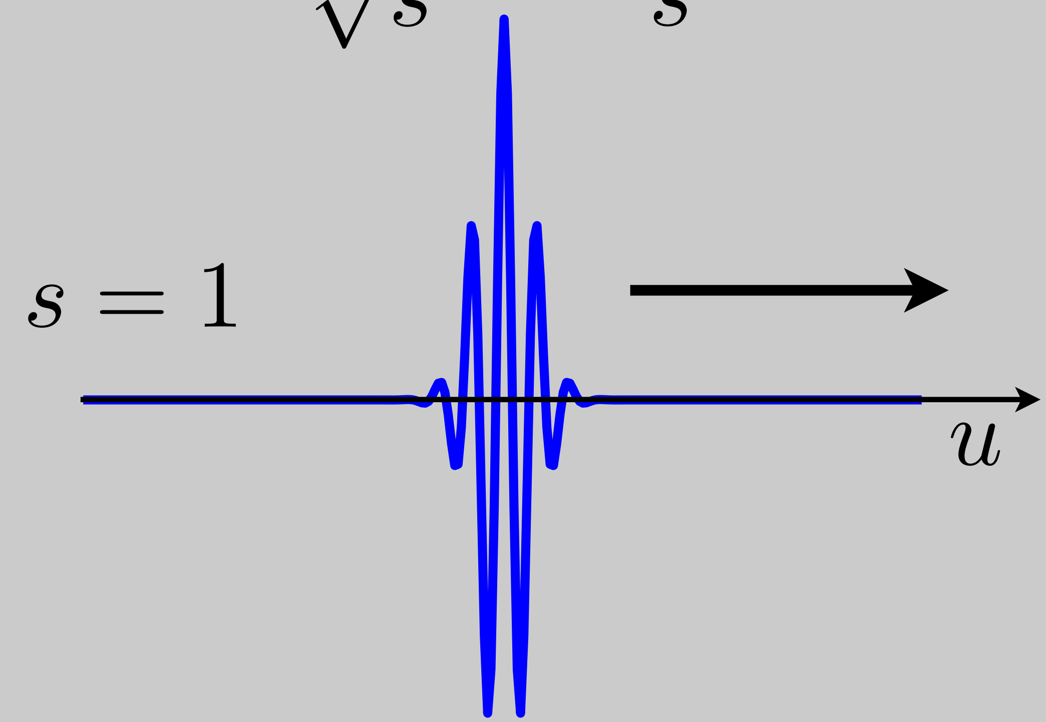
$$w(t-u)e^{j\Omega t}$$

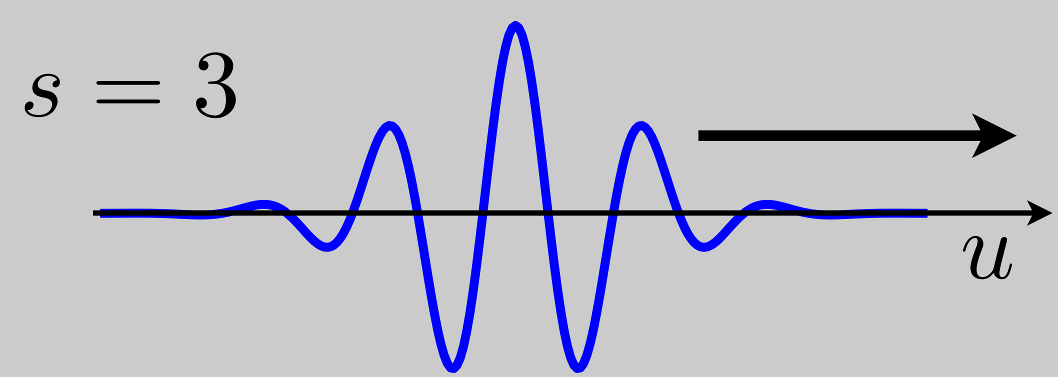




Wavelet Atoms

$$\frac{1}{\sqrt{s}}\Psi\left(\frac{t-u}{s}\right)$$

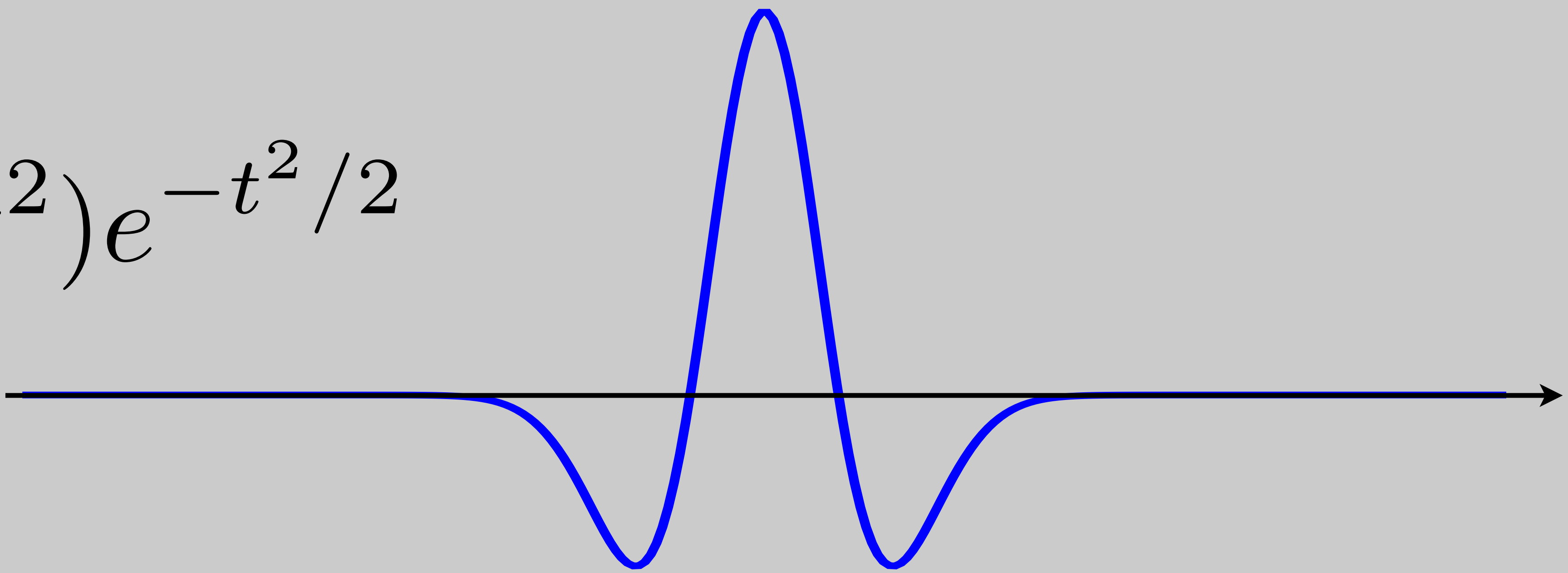




Examples of Wavelets

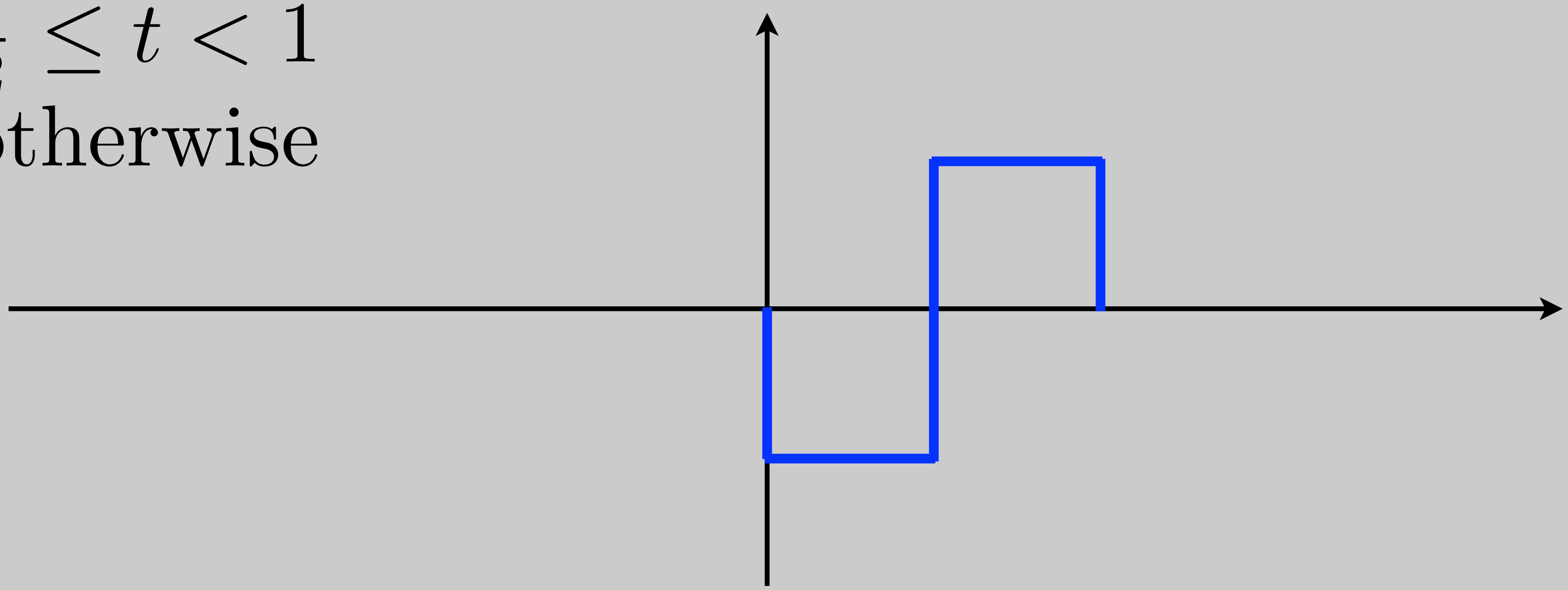
- Mexican Hat

$$\Psi(t) = (1 - t^2)e^{-t^2/2}$$



- Haar

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



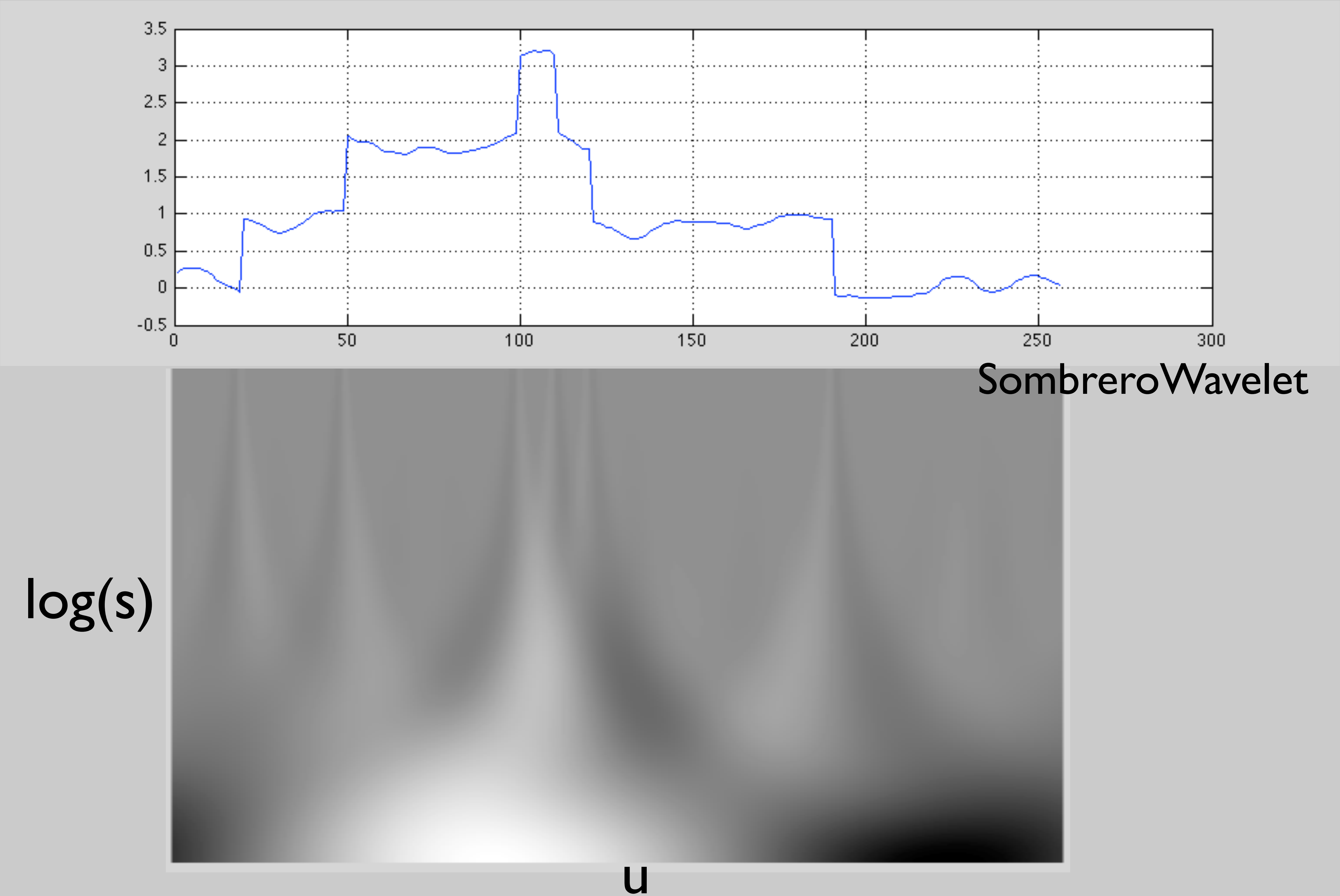
- Can be written as linear filtering

$$
\begin{aligned}
Wf(u,s) &= \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\Psi^*(\frac{t-u}{s})dt \\
&= \left\{f(t) * \overline{\Psi}_s(t)\right\}(u)
\end{aligned}
$$

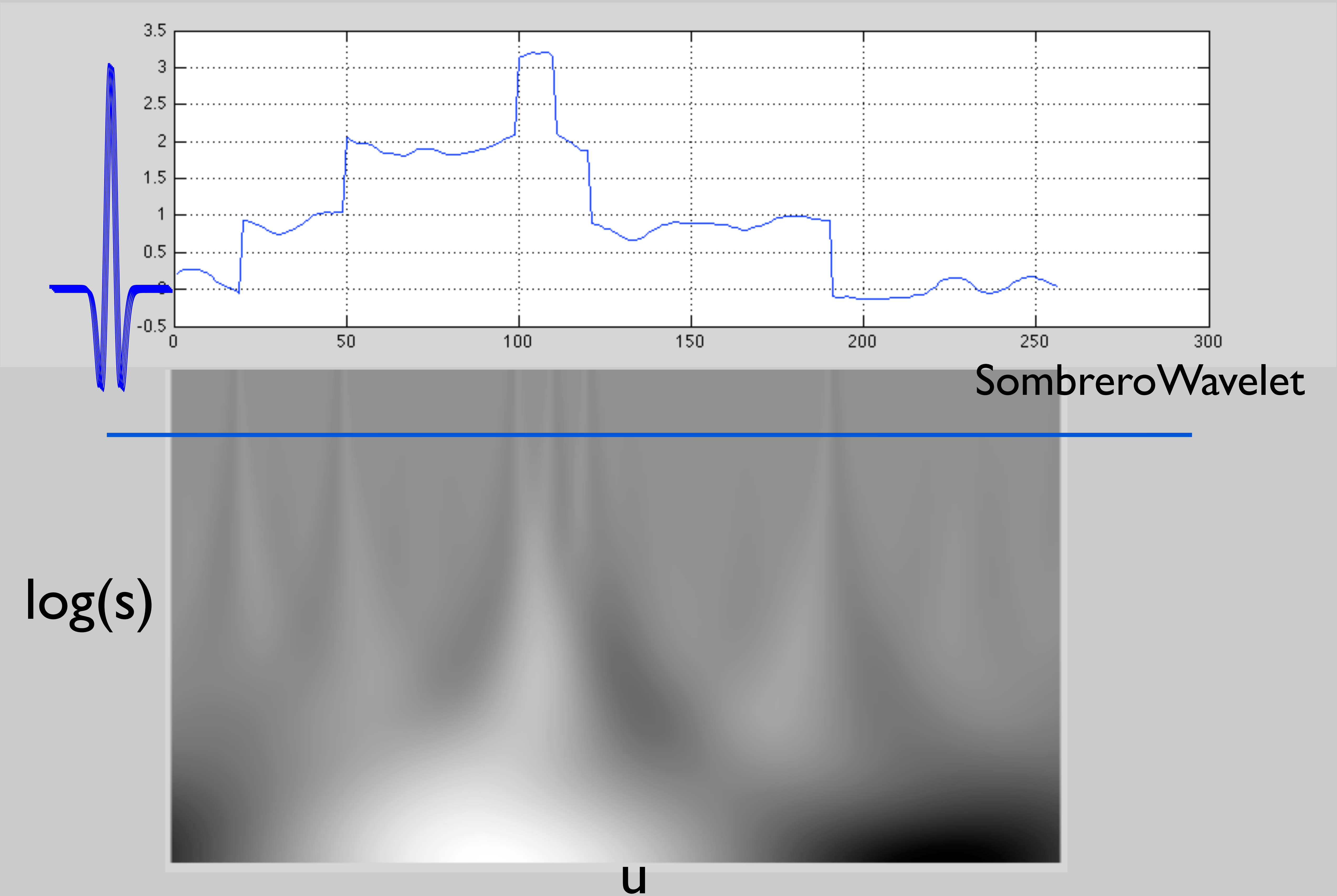
$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

- Wavelet coefficients are a result of bandpass filtering

Example 2: "Bumpy" Signal



Example 2: “Bumpy” Signal



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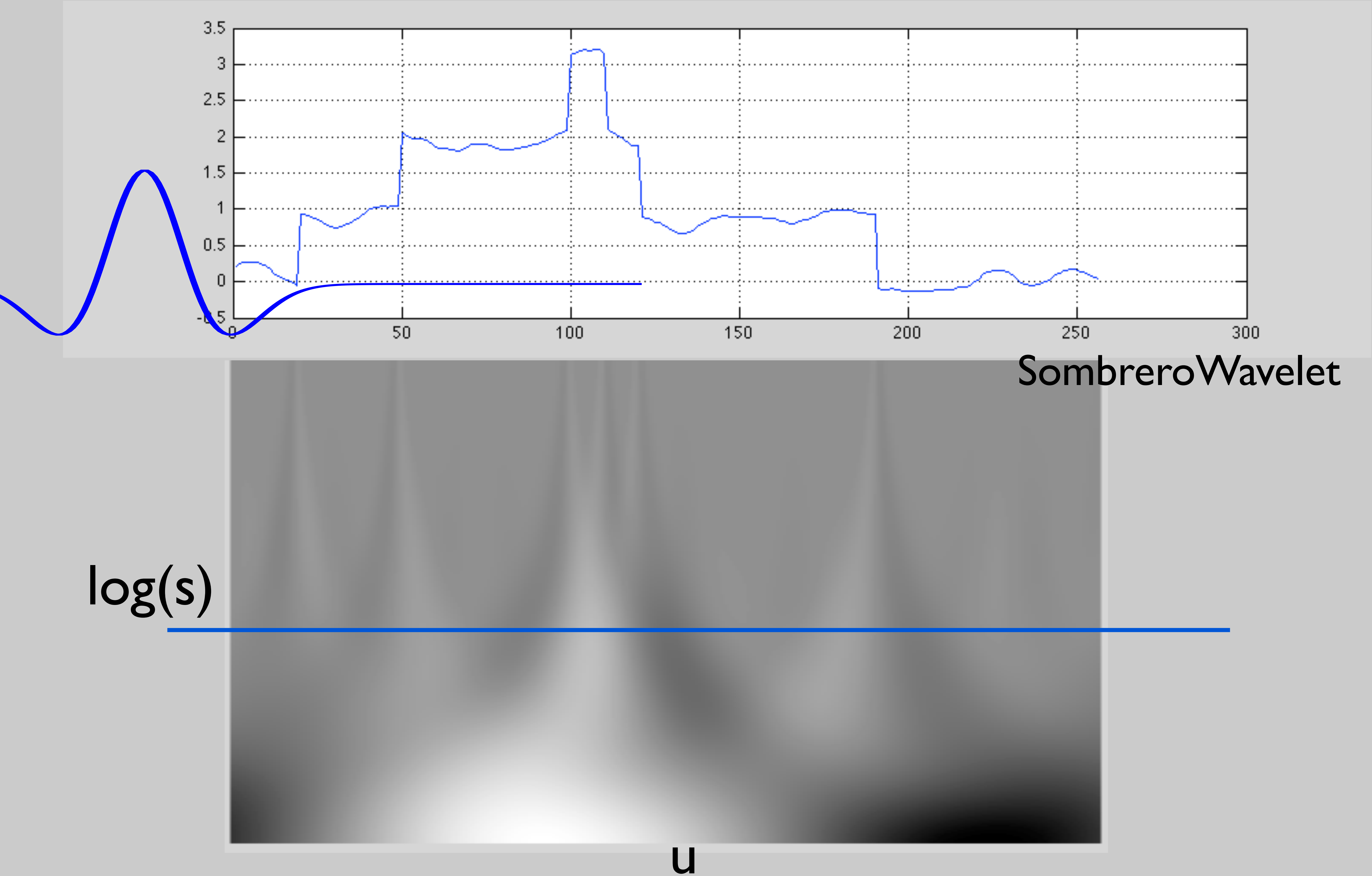

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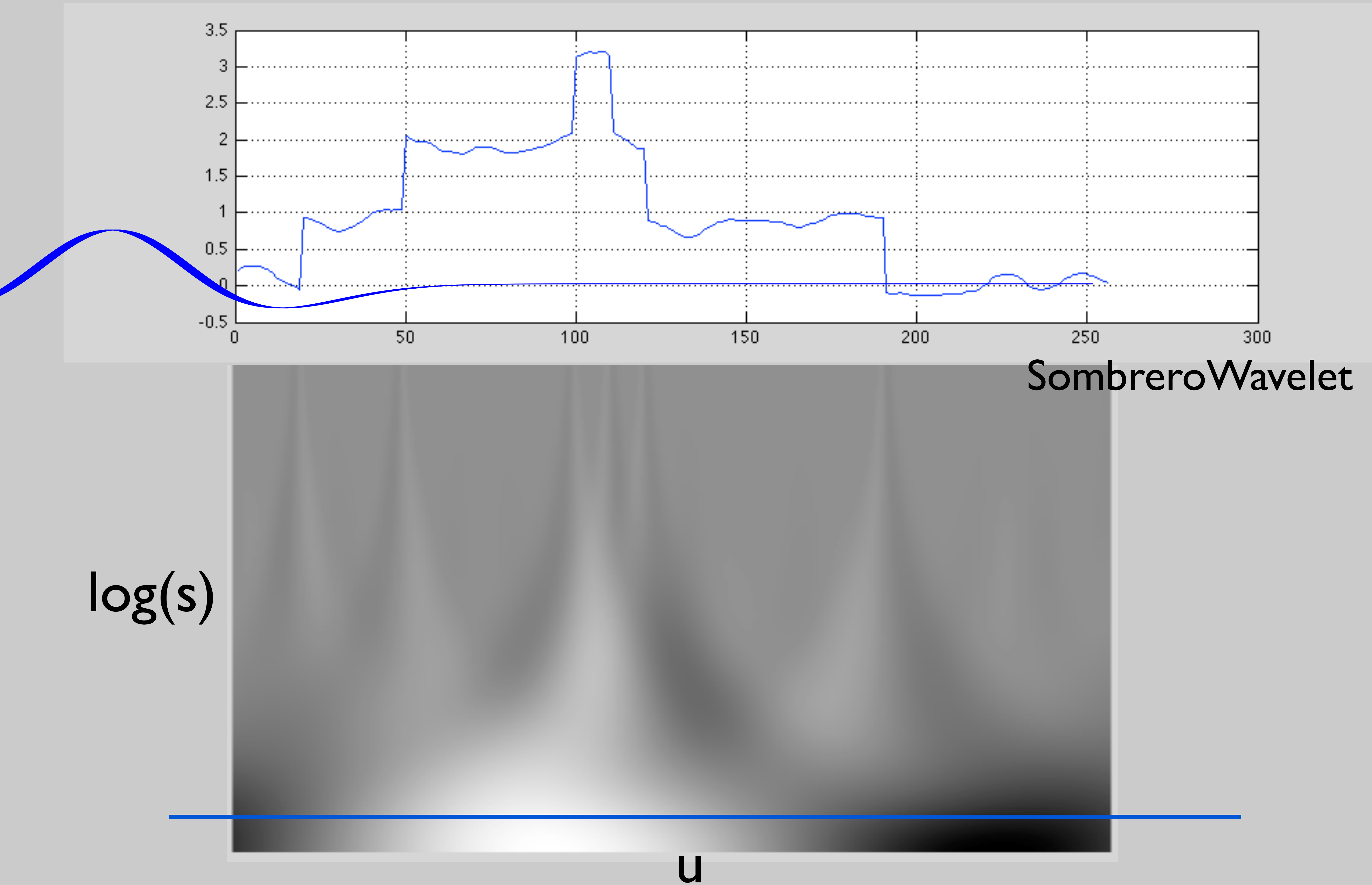


Recall Haar basis fn:

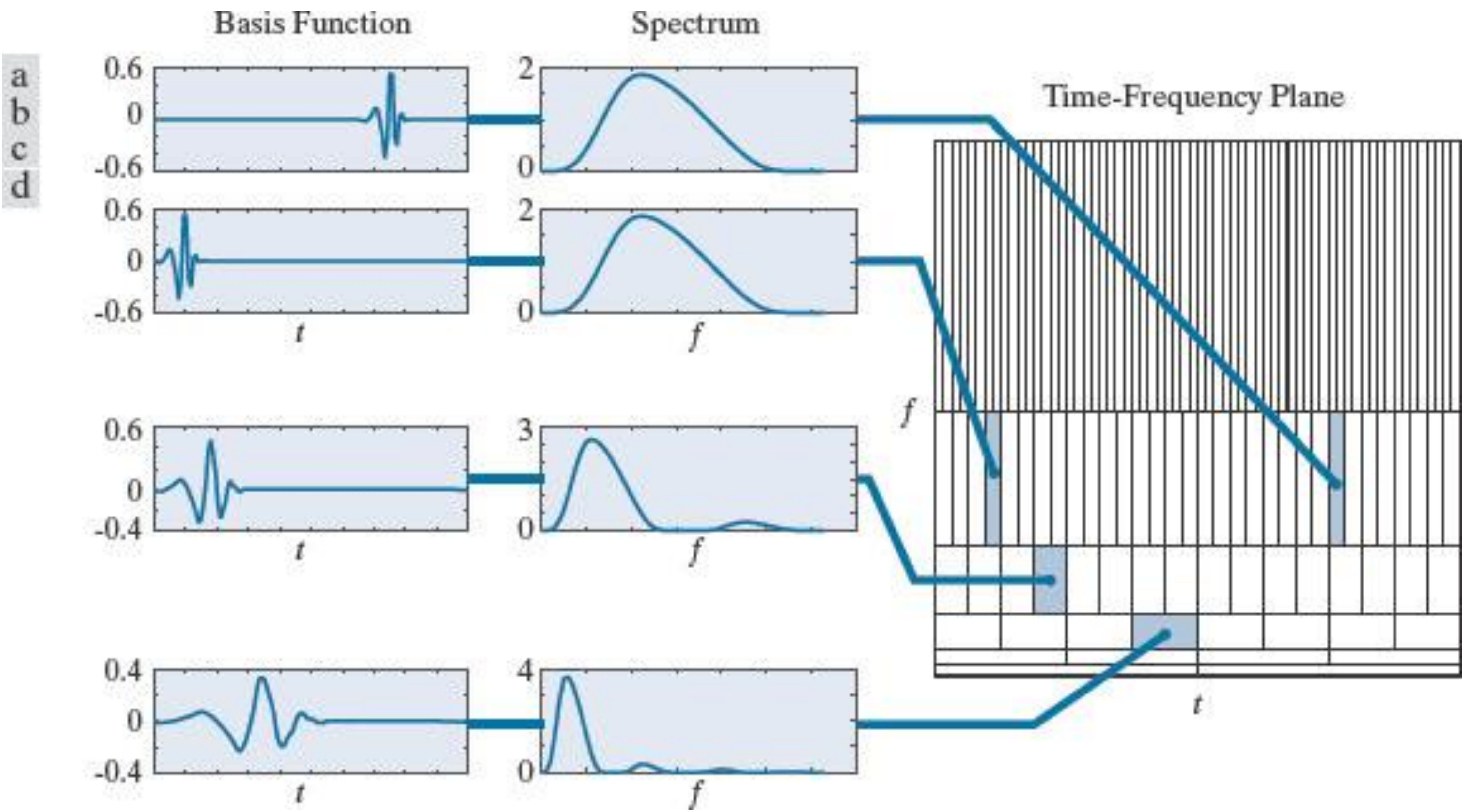
$$\psi_{s,\tau}(t) = 2^{s/2}\,\psi(2^s t - \tau) \qquad \tau, s \text{ integer}$$

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k)$$

$$F.T.\left\{\psi(2^s t)\right\} = \frac{1}{|2^s|}\,\Psi\left(\frac{f}{2^s}\right)$$

for $s > 0$ $\longrightarrow$ spectrum is stretched

for $s < 0$ $\longrightarrow$ spectrum is compressed



Time and frequency localization of 128-point Daubechies basis functions.

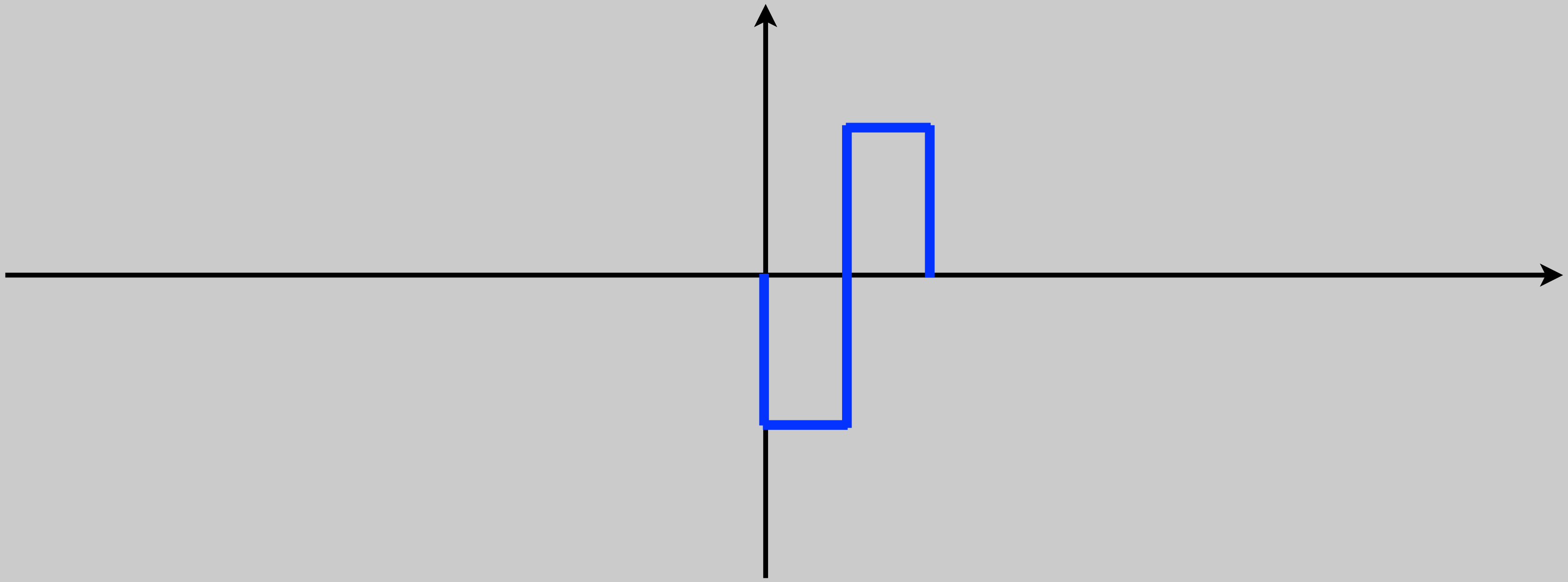
- Many different constructions for different signals
  - –Haar good for piece-wise constant signals
  - –Battle-Lemarie' : Spline polynomials

- Can construct Orthogonal wavelets
  - – For example: dyadic Haar is orthonormal

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$

$$i = [0, 1, 2, \cdots]$$

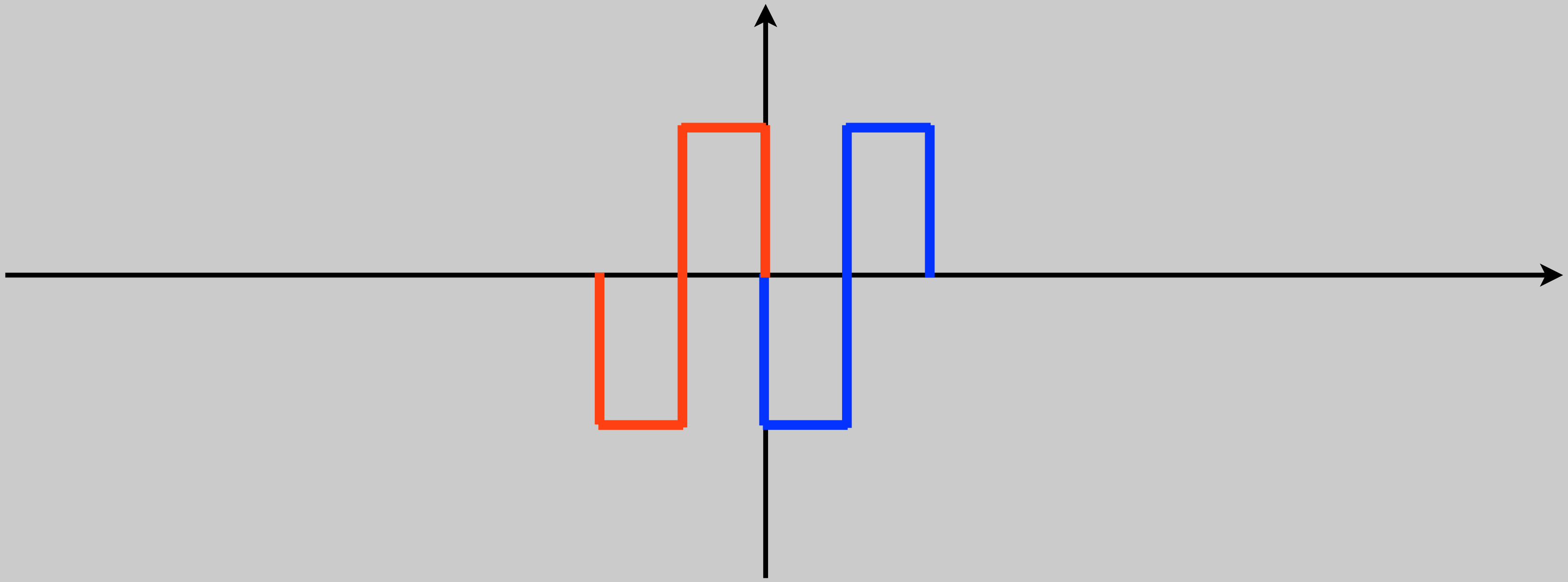
Orthonormal Haar - Basis functions



Same scale
non-overlapping

$$\overline{\Psi}_{0,0}(t) = \frac{1}{\sqrt{2^0}}\Psi(\frac{t-2^0 0}{2^0}) = \Psi(t)$$

Orthonormal Haar - Basis functions

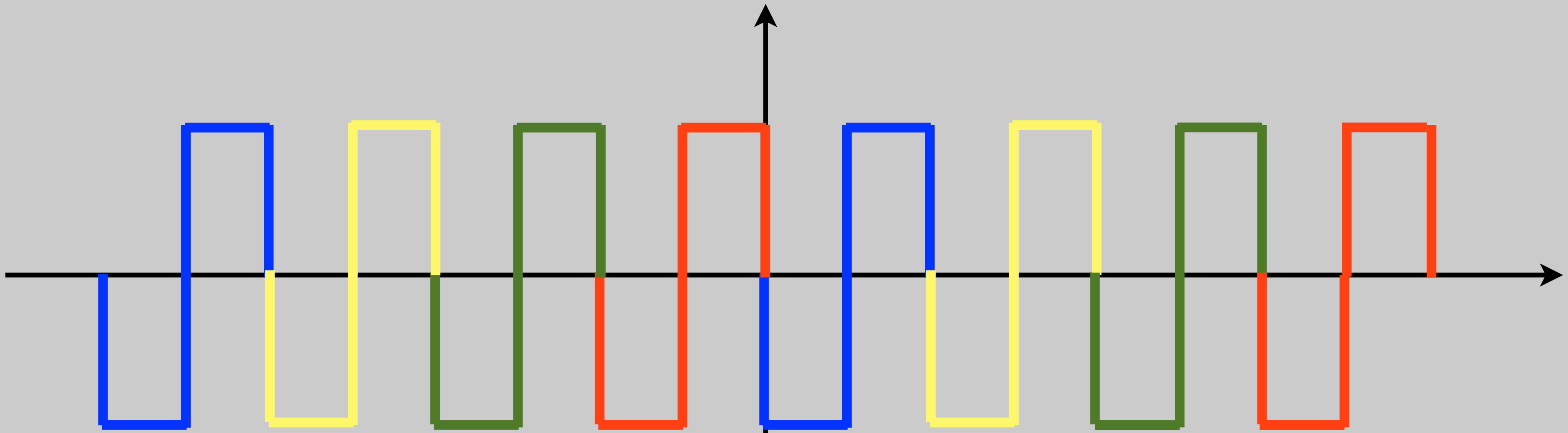


Same scale
non-overlapping

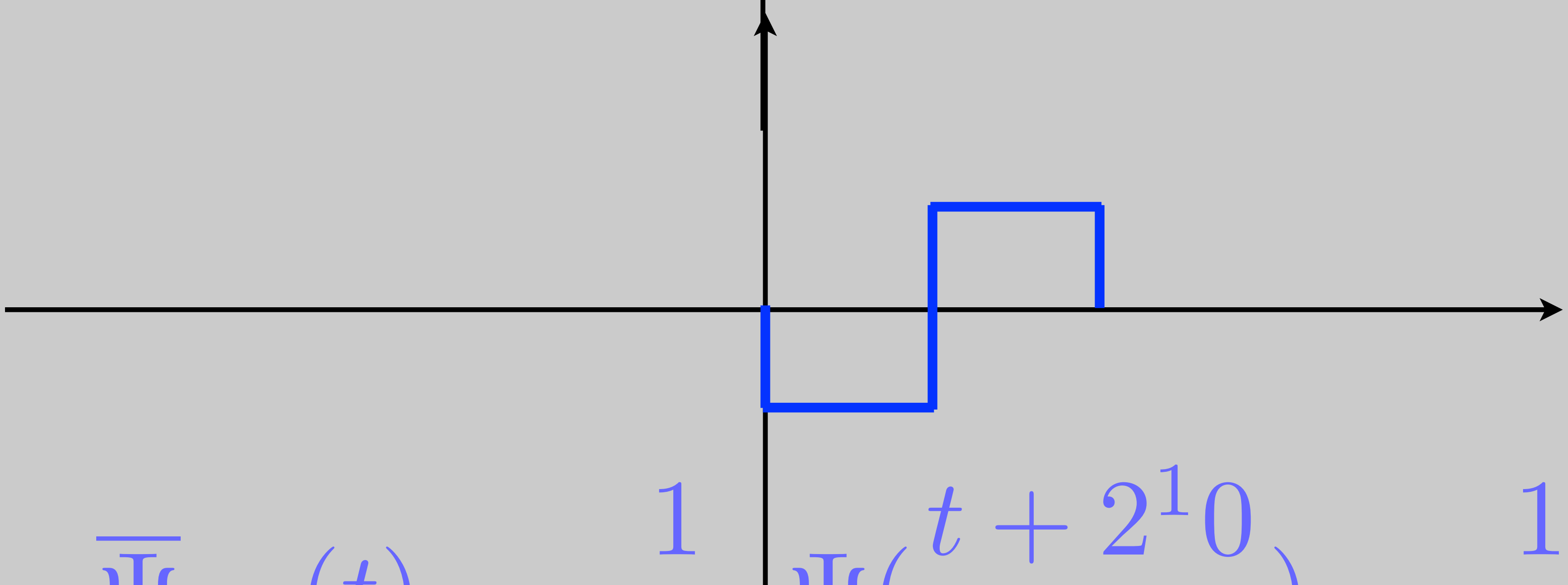
$$\overline{\Psi}_{0,0}(t) = \frac{1}{\sqrt{2^0}}\Psi(\frac{t-2^0 0}{2^0}) = \Psi(t)$$

$$\overline{\Psi}_{0,-1}(t) = \frac{1}{\sqrt{2^0}}\Psi(\frac{t+2^0 1}{2^0}) = \Psi(t+1)$$

Orthonormal Haar



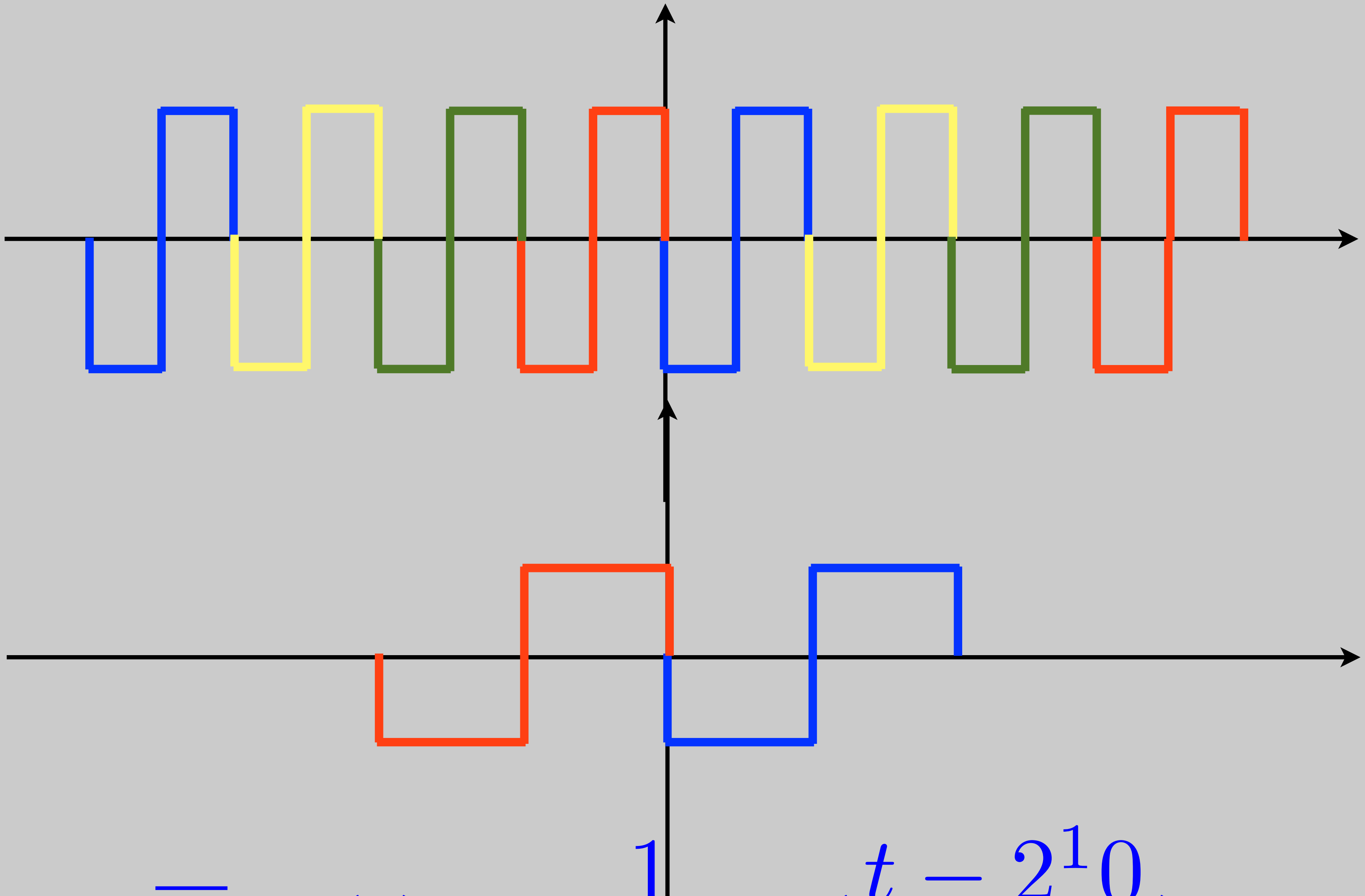
Same scale non-overlapping



Orthogonal between scales

$$\overline{\Psi}_{1,0}(t) = \frac{1}{\sqrt{2^1}}\Psi(\frac{t+2^1 0}{2^1}) = \frac{1}{\sqrt{2}}\Psi(\frac{t}{2})$$

Orthonormal Haar



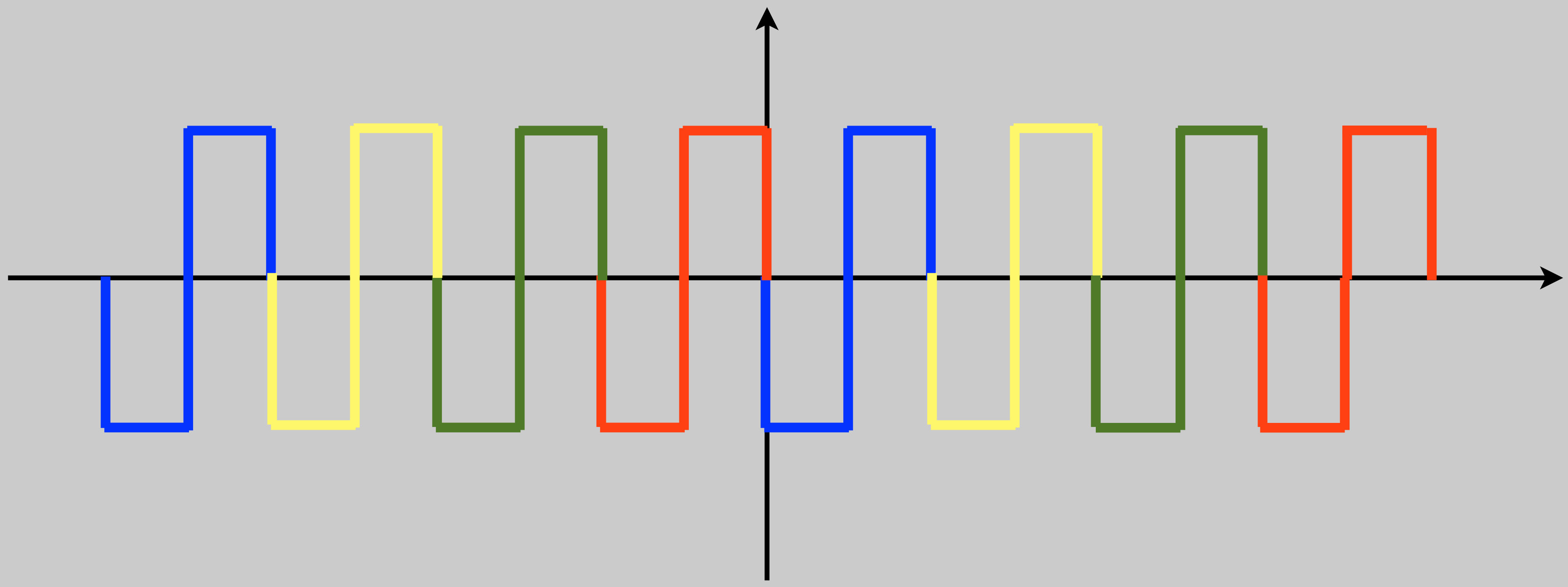
Same scale non-overlapping

Orthogonal between scales

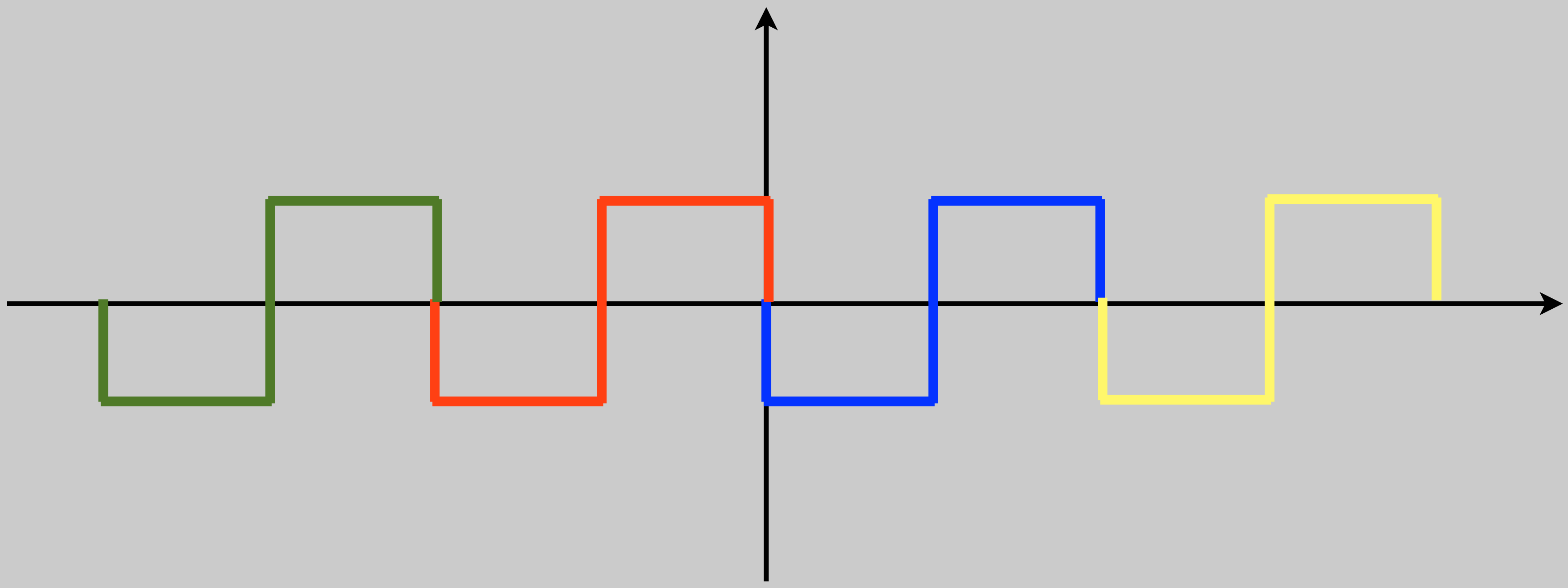
$$\overline{\Psi}_{1,0}(t) = \frac{1}{\sqrt{2^1}}\Psi\left(\frac{t-2^1 0}{2^1}\right) = \frac{1}{\sqrt{2}}\Psi\left(\frac{t}{2}\right)$$

$$\overline{\Psi}_{1,-1}(t) = \frac{1}{\sqrt{2^1}}\Psi\left(\frac{t+2^1 1}{2^1}\right) = \frac{1}{\sqrt{2}}\Psi\left(\frac{t+2}{2}\right)$$

Orthonormal Haar



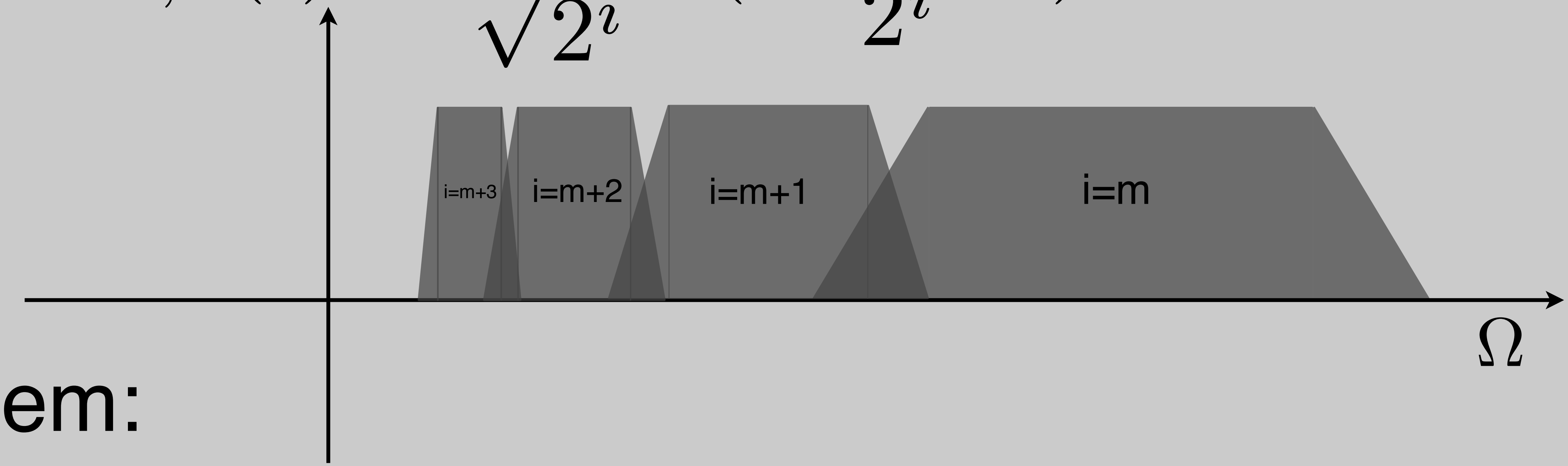
Same scale
non-overlapping



Orthogonal
between scales

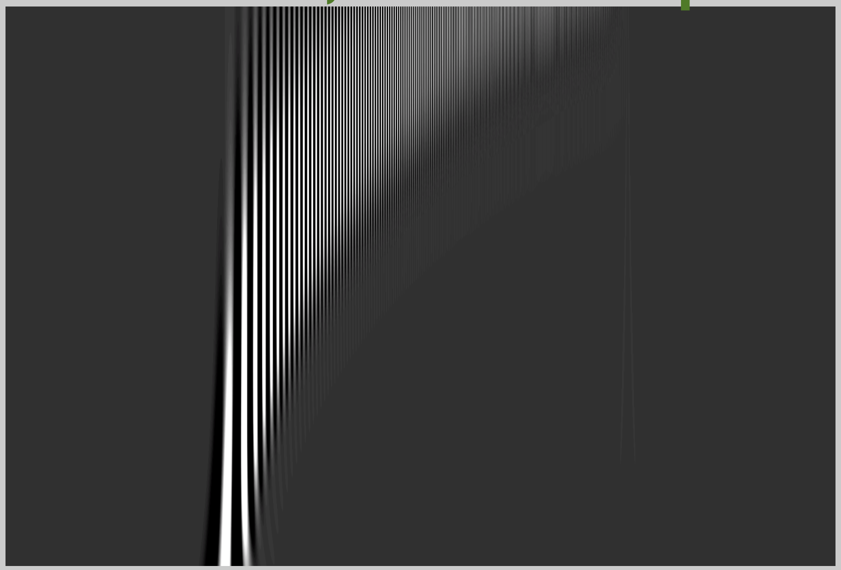
Scaling function

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



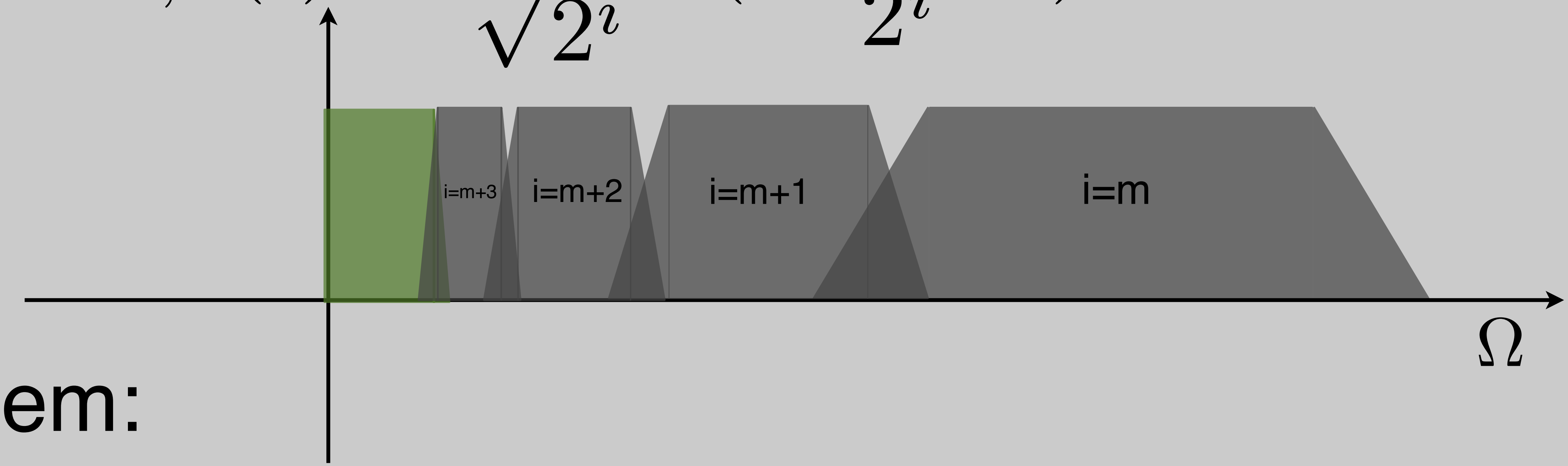
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions

recall, for chirp:



Scaling function

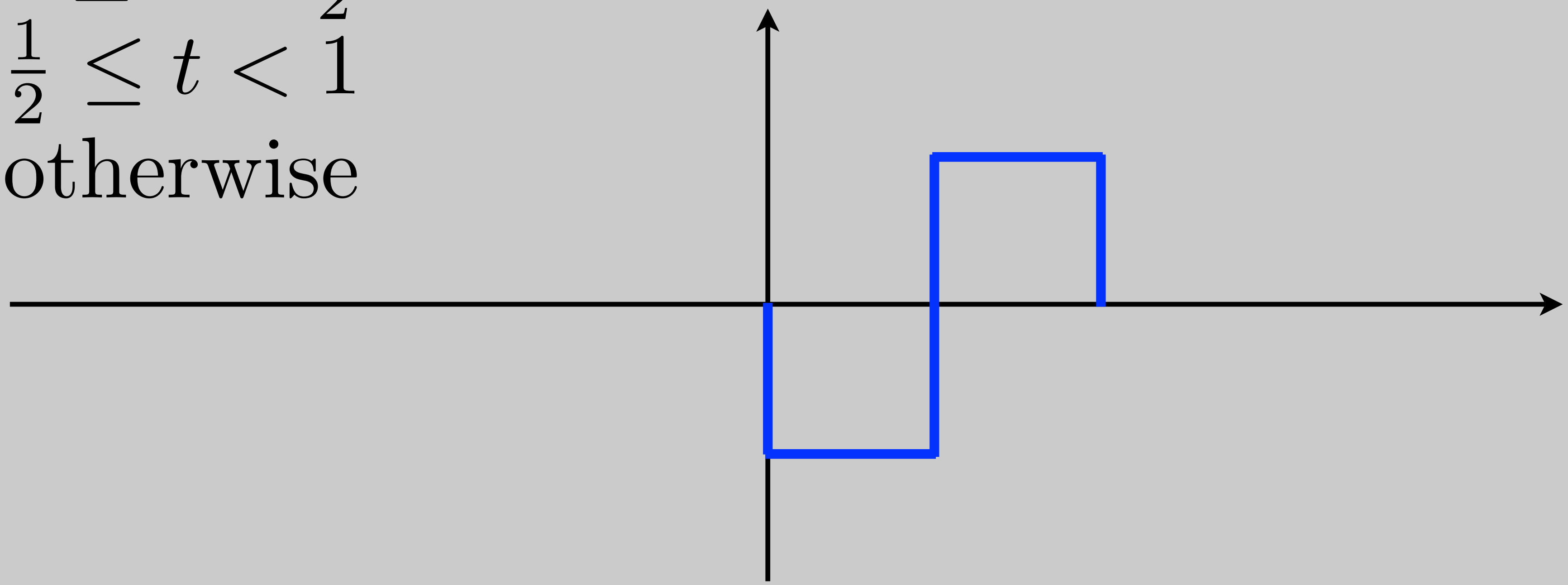
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



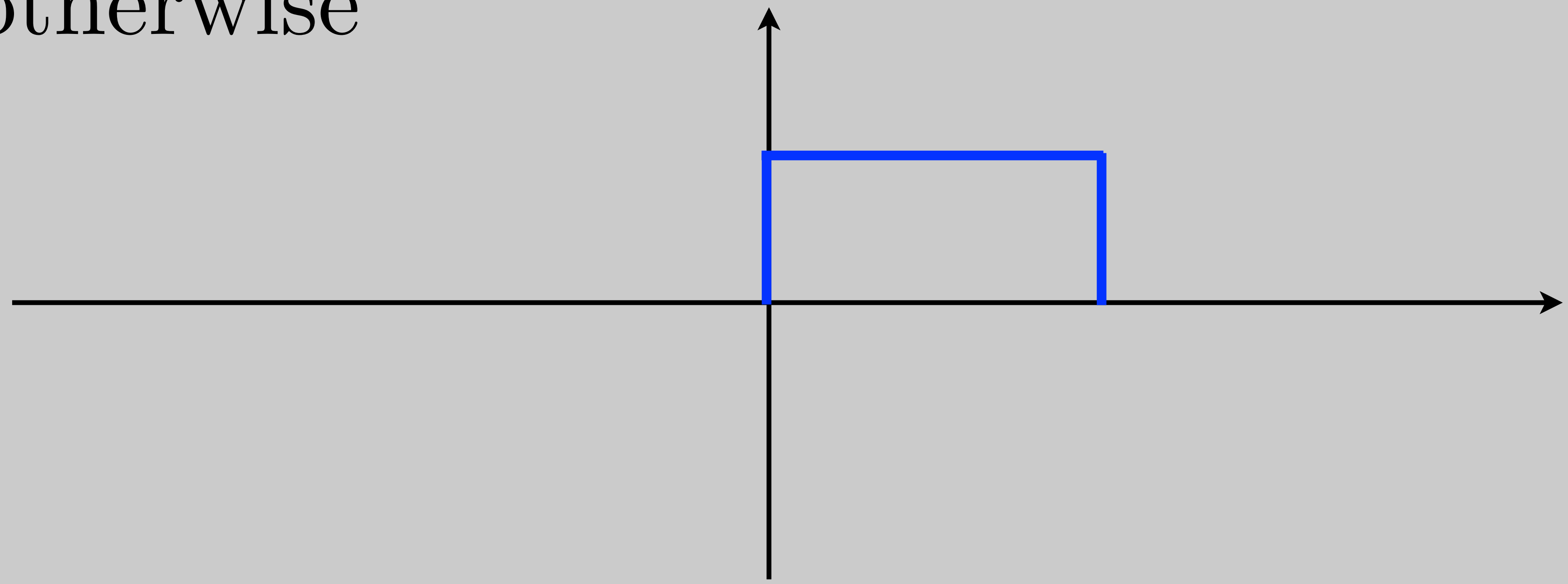
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions
- Solution:
  - –Plug low-pass spectrum with a scaling function $\overline{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \leq t < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Back to Discrete

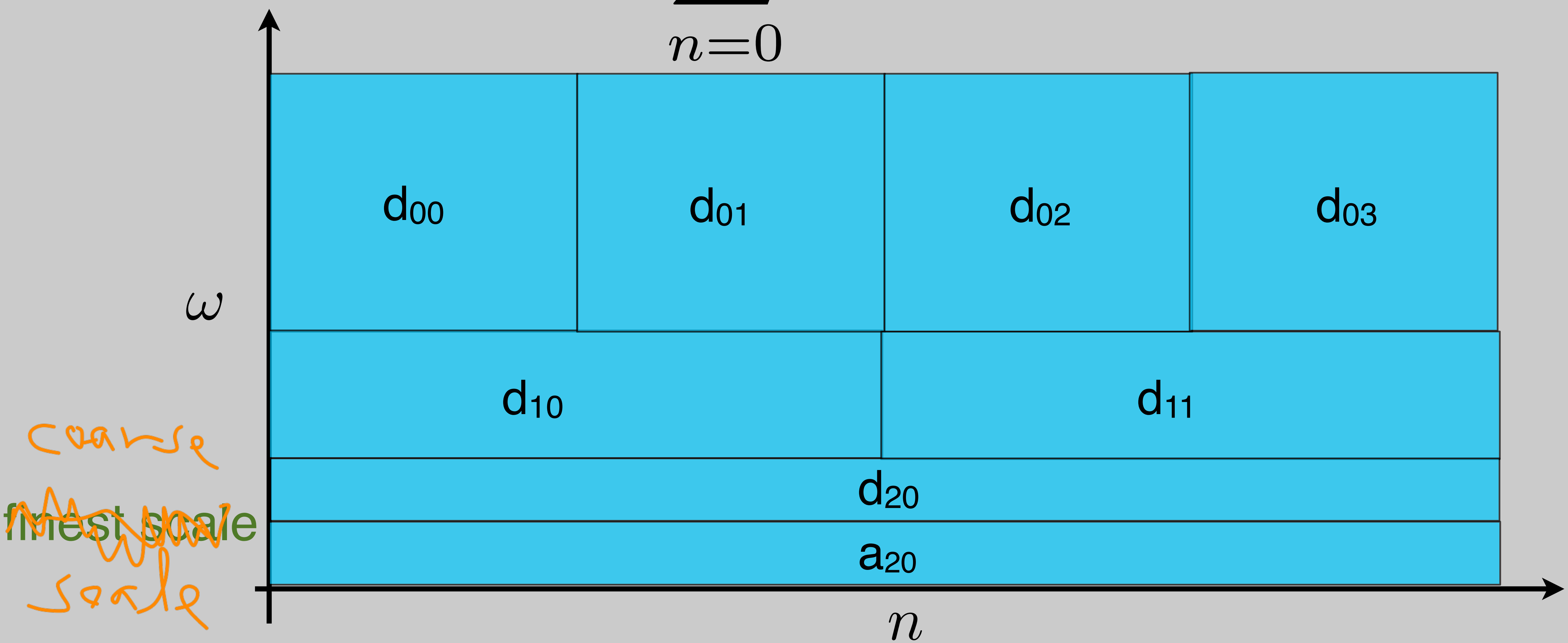
- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.

- From CWT to DWT not so trivial!
- Must take care to maintain properties

Discrete Wavelet Transform

$$d_{s,u} = \sum_{n=0}^{N-1} x[n]\Psi_{s,u}[n]$$

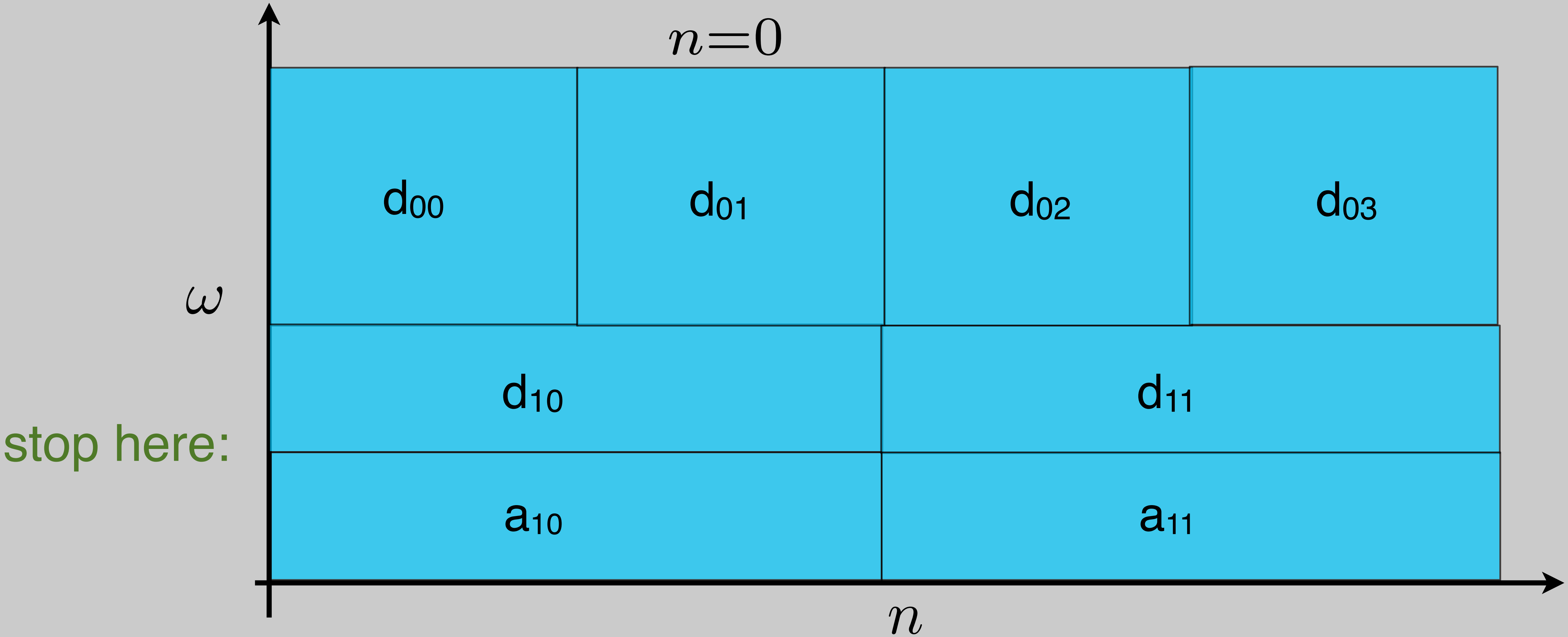
$$a_{s,u} = \sum_{n=0}^{N-1} x[n]\Phi_{s,u}[n]$$



Discrete Wavelet Transform

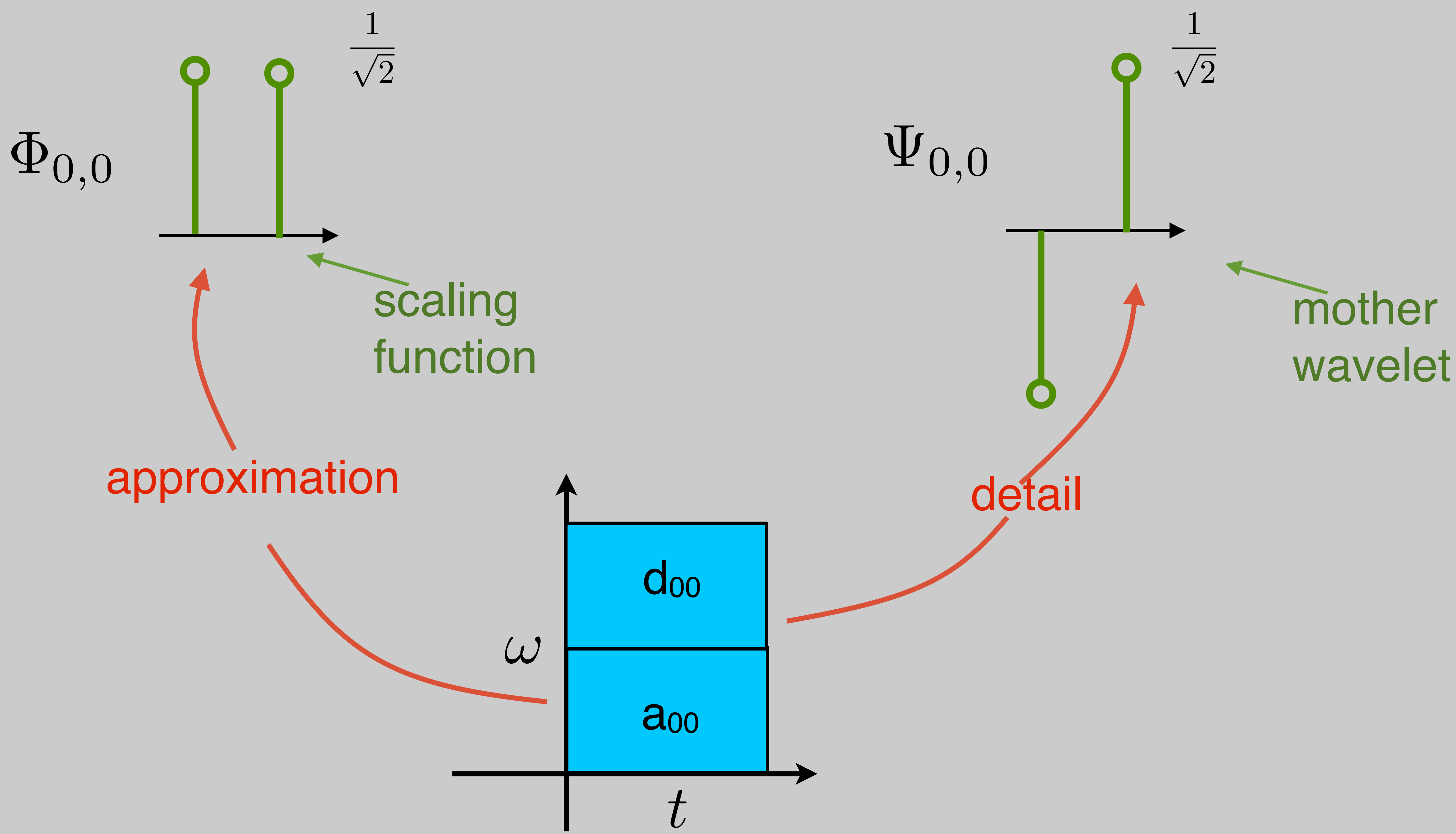
$$d_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Psi_{\mathrm{s,u}}[n]$$

$$a_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Phi_{\mathrm{s,u}}[n]$$



Example: Discrete Haar Wavelet

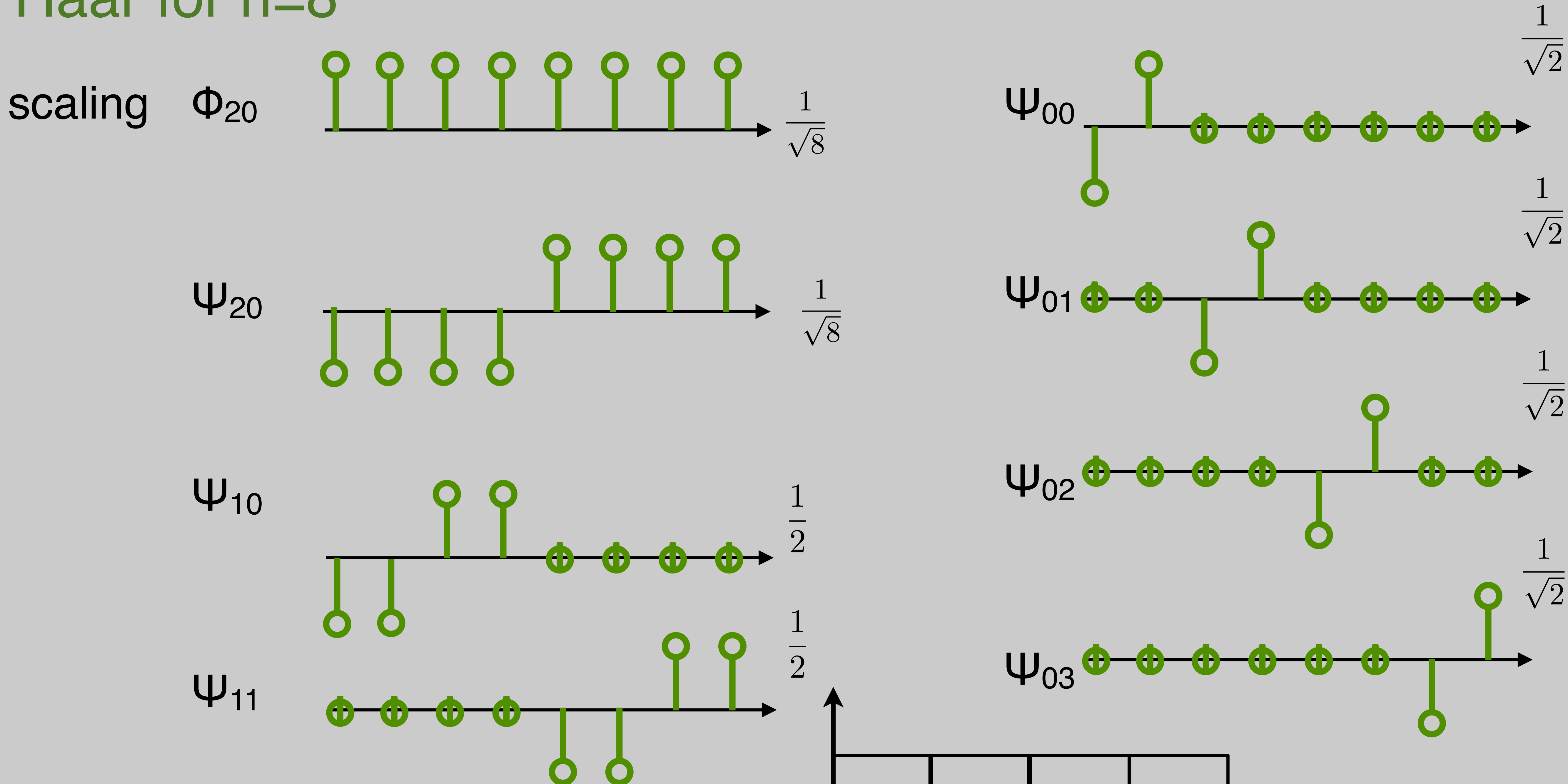
Haar for n=2

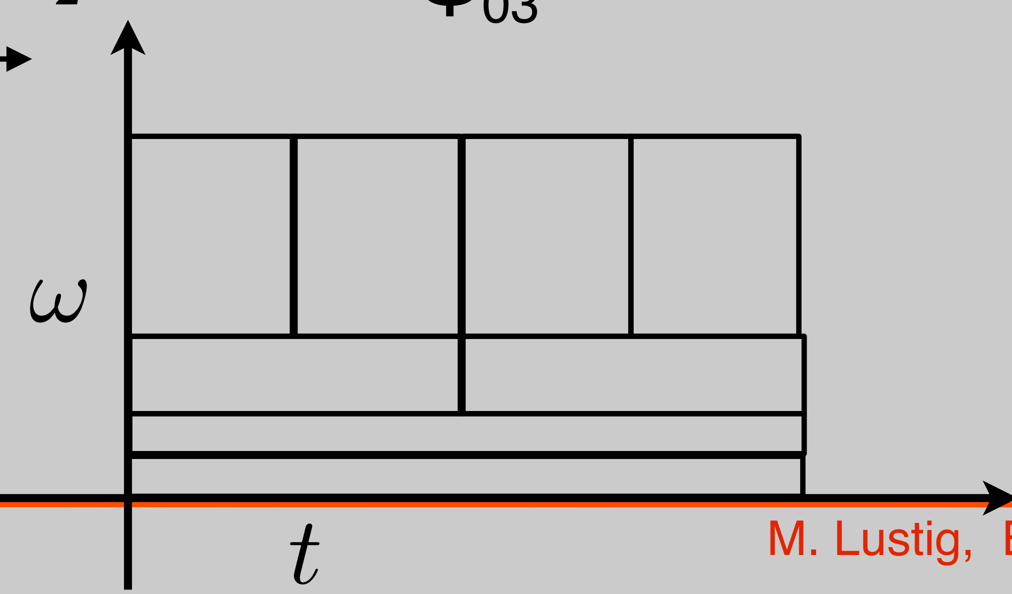


Equivalent to DFT_2!

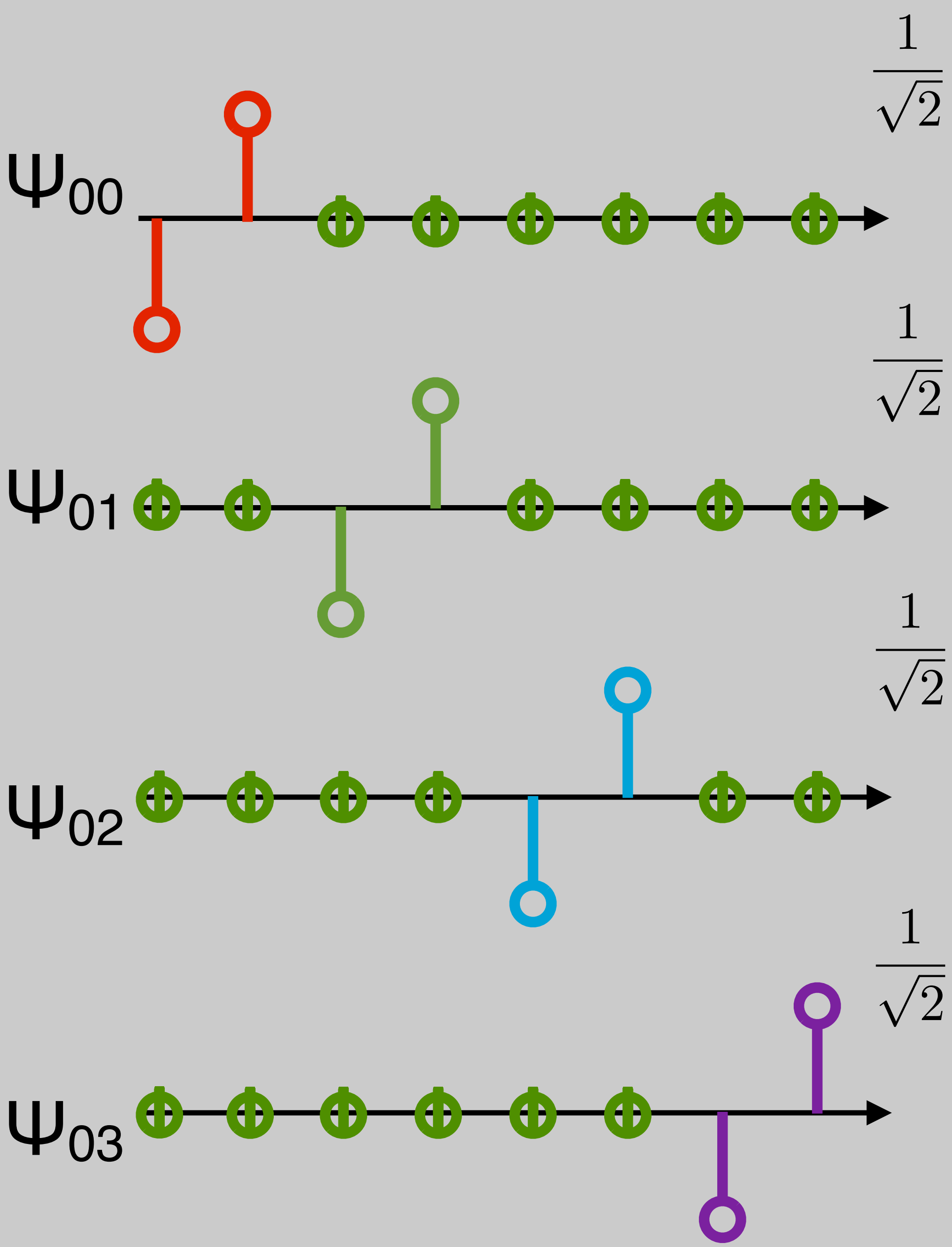
Discrete Orthogonal Haar Wavelet

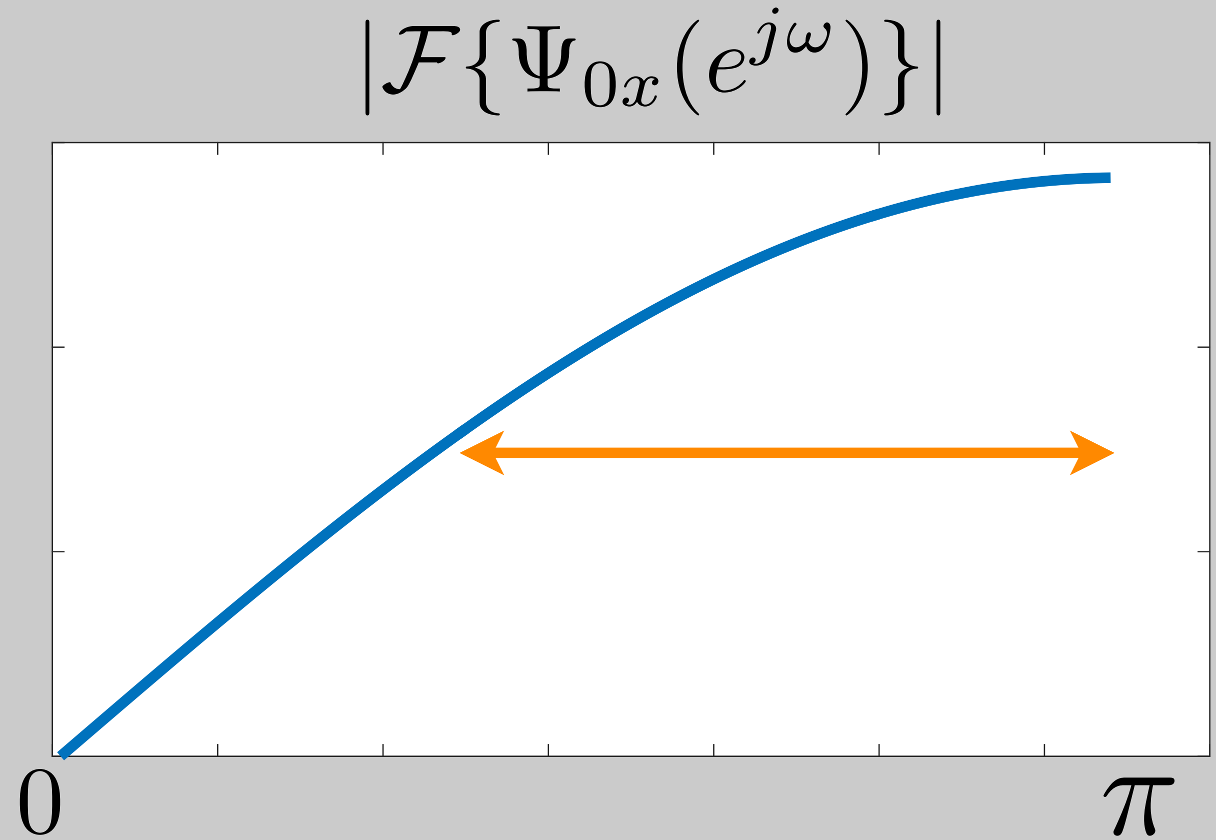
Haar for n=8

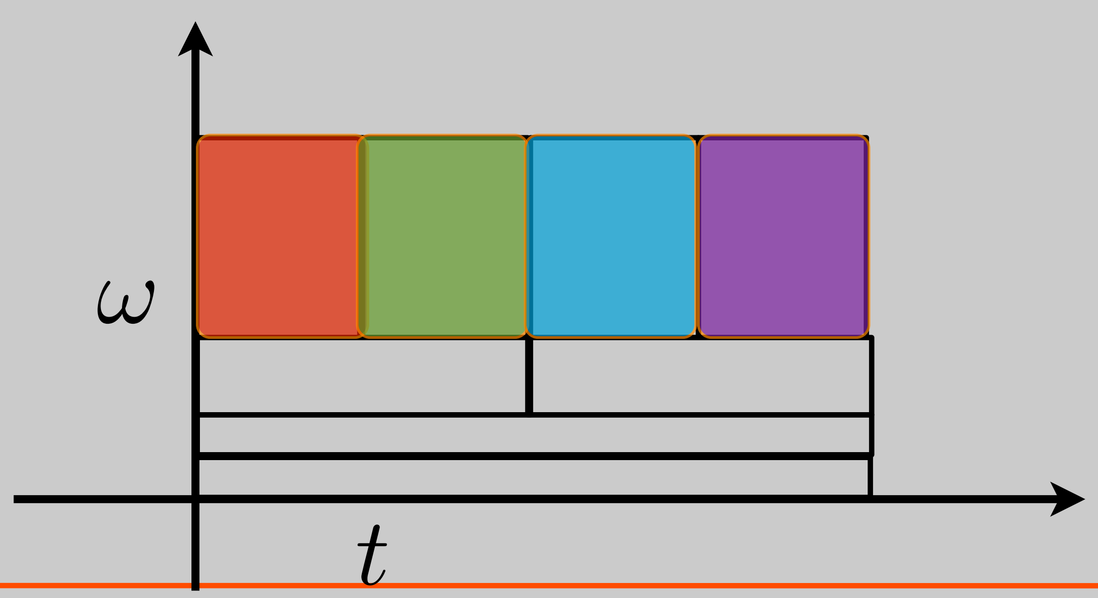




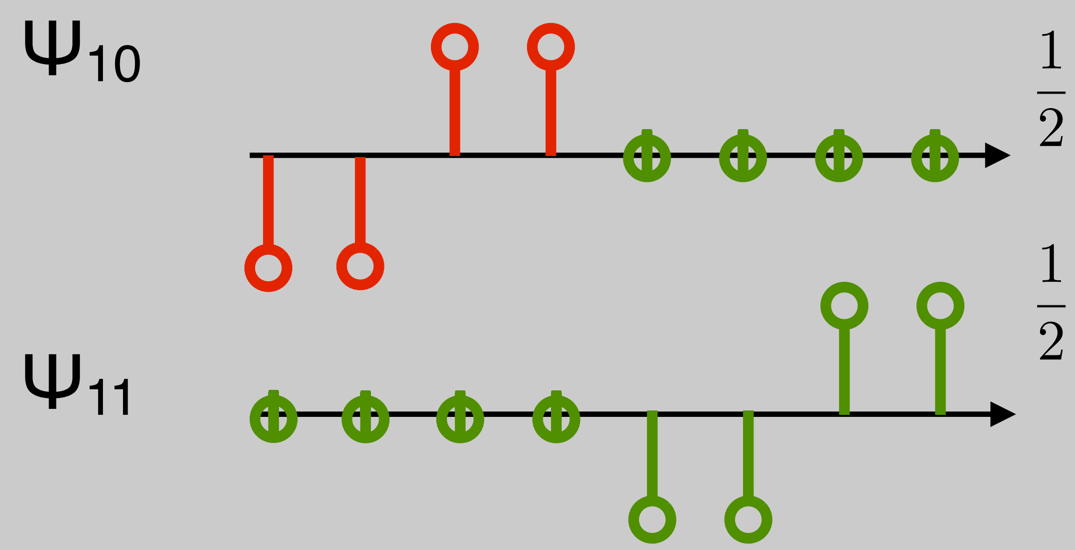
Discrete Orthogonal Haar Wavelet

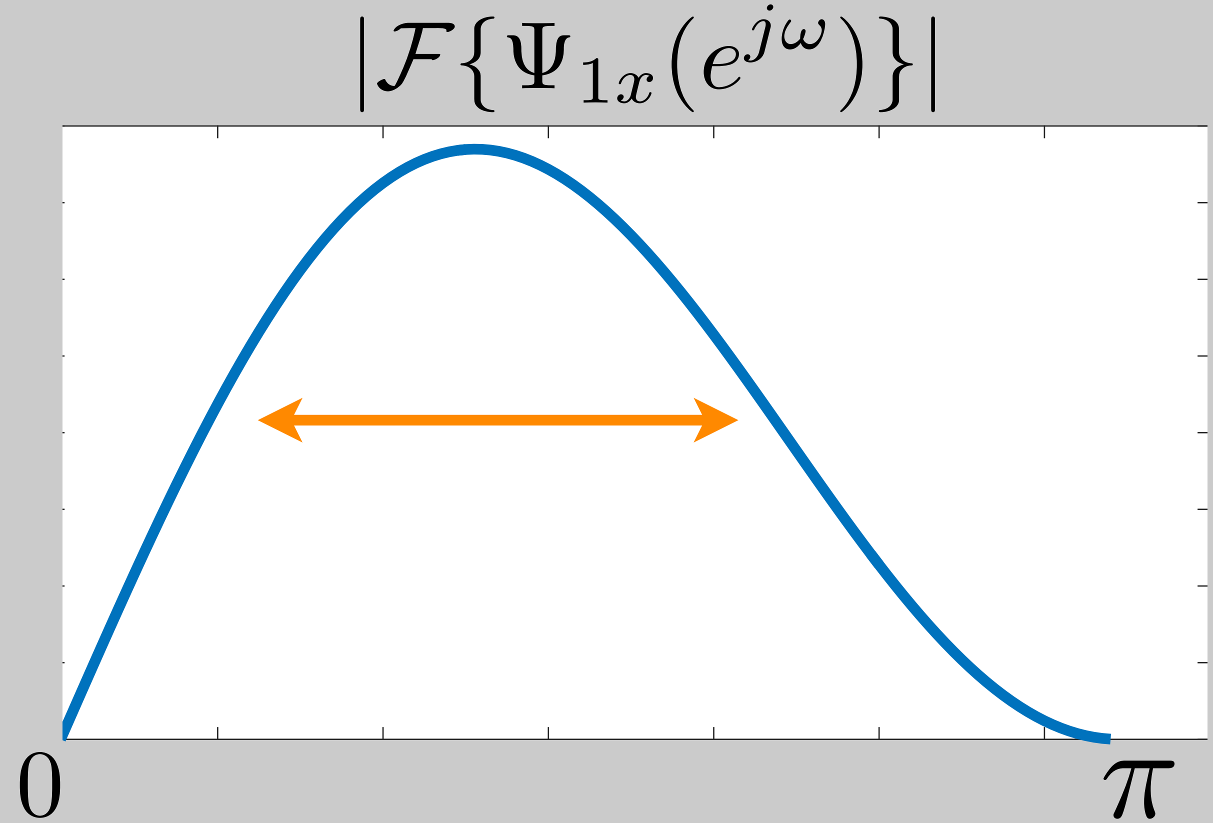


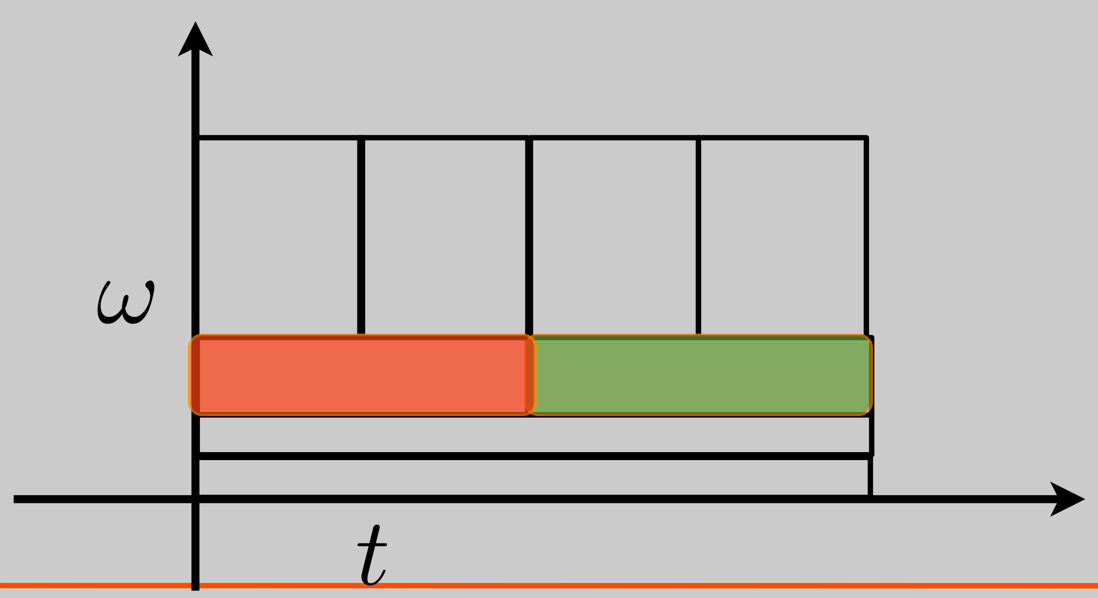




Discrete Orthogonal Haar Wavelet

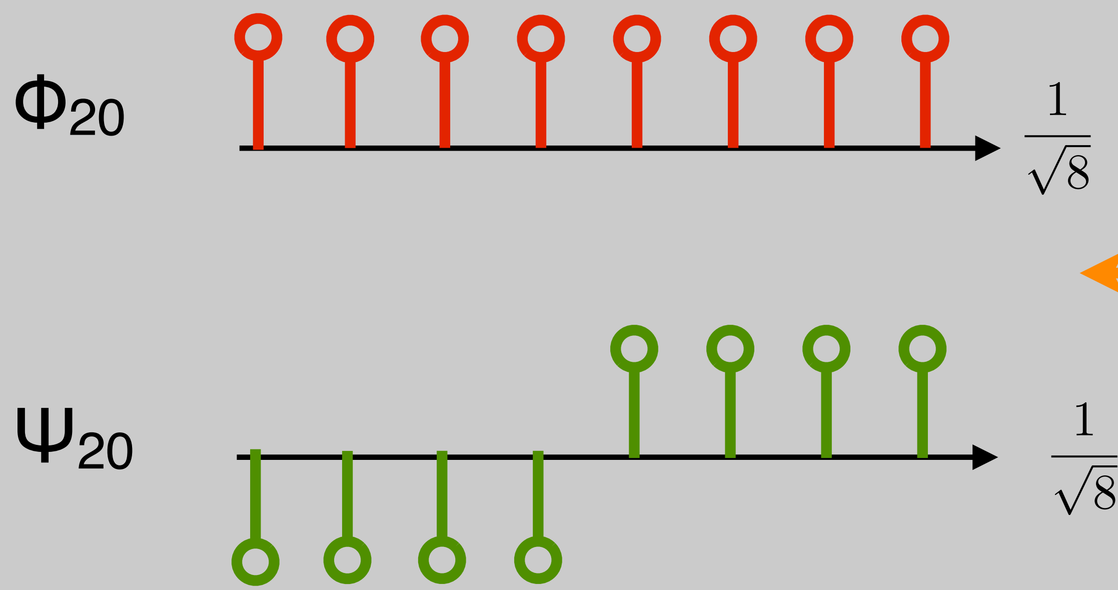


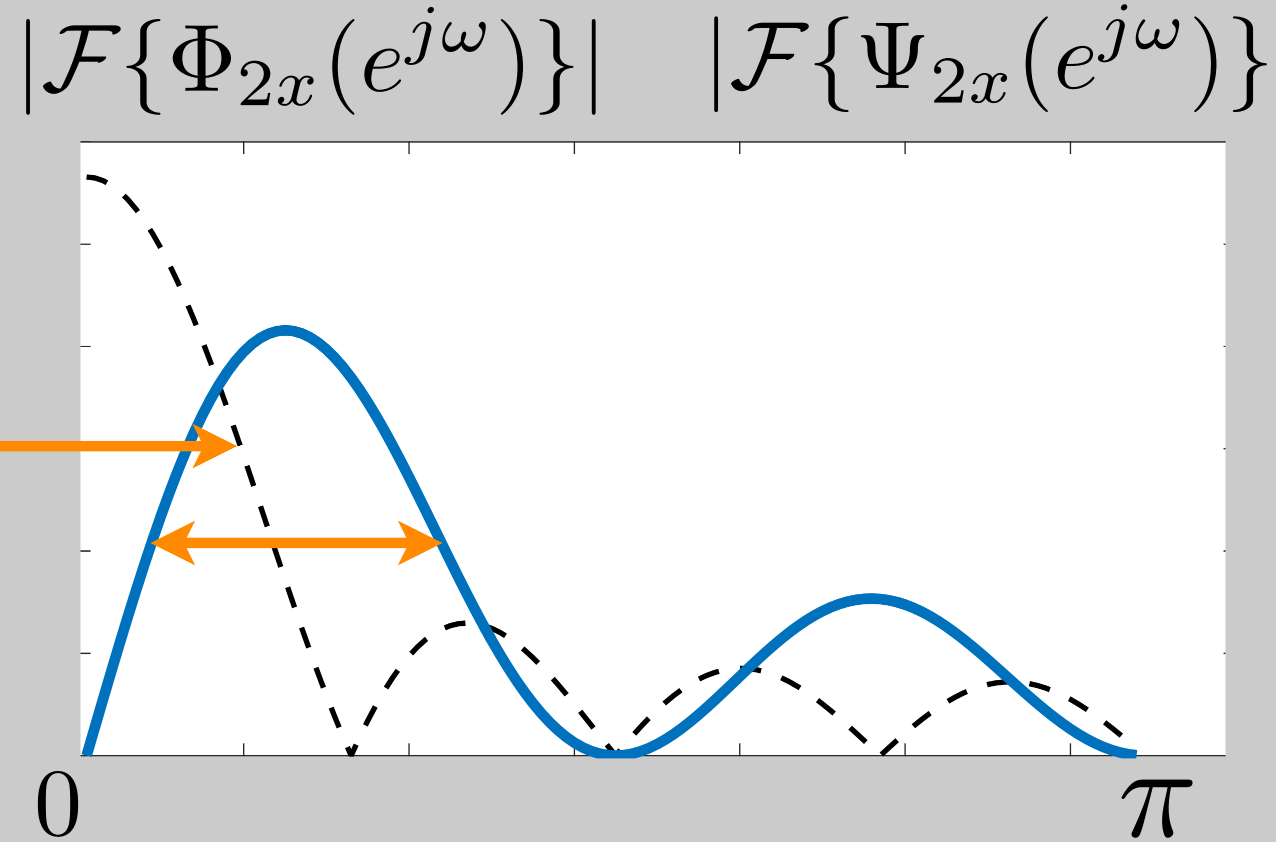


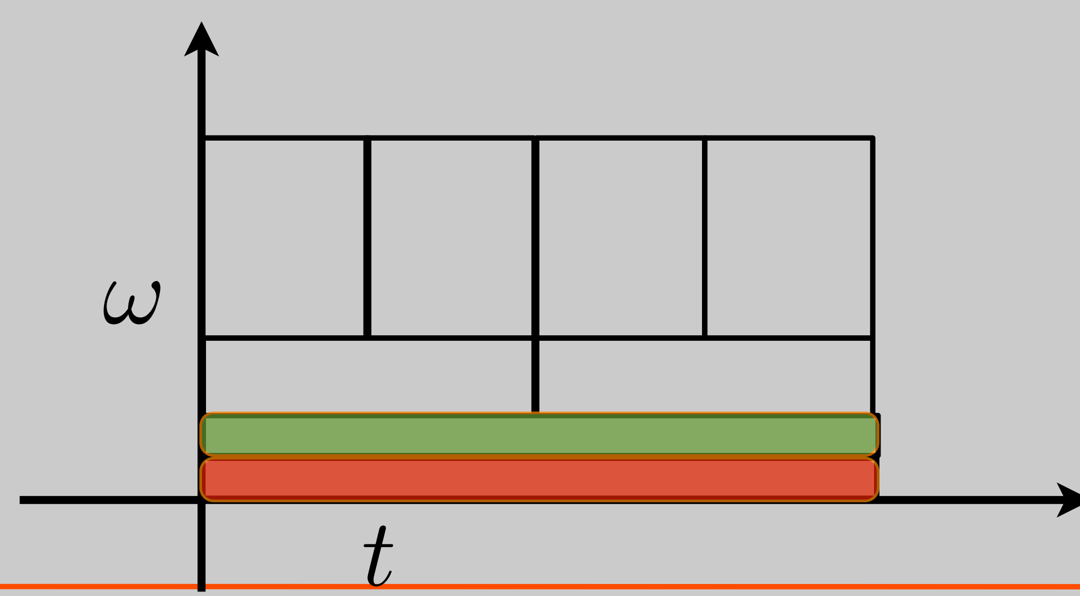


Discrete Orthogonal Haar Wavelet

scaling







Optional: stop decomposition at Level 1

scaling