Now that you are familiar with generating phase portraits, please feel free to use tools to help you generate these automatically. In particular, there is a nice set of tools, pplane, developed for MATLAB by Professor Polking at Rice University. The web link is on the course web site.

**Problem 1: Degenerate phase portraits.** Consider the second-order linear system
\[ \dot{x} = Ax, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \]
and \( A \in \mathbb{R}^{2 \times 2} \). (Note that \( x_1 \) and \( x_2 \) are general variables in the system, i.e. \( x_2 \) is not necessarily equal to \( \dot{x}_1 \)). Let \( \lambda_1, \lambda_2 \) be the eigenvalues of \( A \). Analyze the situations (i) \( \lambda_1 = 0, \lambda_2 < 0 \); (ii) \( \lambda_1 = 0, \lambda_2 = 0 \), by determining the equilibria in each case, sketching phase plots, and explaining your results.

**Problem 2: Modification of Duffing’s equation [4].** Consider the modified Duffing equation
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - x_1^3 - \delta x_2 + x_1^2 x_2
\end{align*}
\]
Find its equilibria. Linearize about the equilibria. Apply Bendixson’s theorem to rule out regions of limit cycles. Synthesize this information to conjecture plausible phase portraits of the system.

**Problem 3: First Integrals.** One way of studying differential equations in the plane
\[ \dot{x} = f(x), \quad x \in \mathbb{R}^2 \]
is to attempt to find scalar functions \( V : \mathbb{R}^2 \to \mathbb{R} \) such that
\[ \frac{d}{dt} V(x) = \frac{\partial V}{\partial x}(x) \cdot f(x) = 0 \]
meaning that \( V(x) \) is constant along trajectories of (4). Such a function \( V(x) \) is called a first integral of the motion of the system.

Now consider the nonlinear differential equation
\[ \ddot{\theta} = 1 - 2 \sin \theta \]
Determine the equilibria of this system and their stability type. In particular, show that some of the equilibria correspond to nonlinear centers, by finding a first integral for this system. SKETCH an approximate phase portrait for (6).

**Problem 4: Hamiltonian Systems.**
A Hamiltonian system is one in which
\[
\begin{align*}
\dot{x}_1 &= \frac{\partial H(x_1, x_2)}{\partial x_2} \\
\dot{x}_2 &= -\frac{\partial H(x_1, x_2)}{\partial x_1}
\end{align*}
\]
for some Hamiltonian function \( H(x_1, x_2) \).
Consider the Duffing equation with $\delta = 0$:

$$\ddot{x} - x + x^3 = 0 \quad (9)$$

Show that for this system (with $x_1 = x$, $x_2 = \dot{x}$), the linearization around the equilibria $(-1,0)$ and $(1,0)$ cannot predict the behavior of the nonlinear system around these equilibria. Even so, if we simulate this nonlinear system, we observe what look like closed orbits around these equilibria. Prove that $(-1,0)$ and $(1,0)$ are in fact centers for the nonlinear system (9), by recognizing that (9) is Hamiltonian, and determining the Hamiltonian function $H(x_1, x_2)$. (HINT: Determine $\dot{H}(x_1, x_2)$ along trajectories of the system.)

**Problem 5: The method of isoclines [2].** A simple but useful technique for the approximation of solution curves in the phase plane is provided by the method of isoclines. Given the nonlinear system

$$\frac{dx_1}{dt} = f_1(x_1, x_2) \quad (10)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2) \quad (11)$$

we can write (assume for the moment that $f_1(x_1, x_2) \neq 0$)

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \quad (12)$$

We seek curves $x_2 = h(x_1)$ on which the slope $dx_2/dx_1 = c$ is constant. Such curves, called isoclines, are given by solving the equation

$$f_2(x_1, x_2) = cf_1(x_1, x_2) \quad (13)$$

Now consider the following system:

$$\dot{x}_1 = x_2^2 - x_1x_2 \quad (14)$$
$$\dot{x}_2 = -x_2 + x_1^2 \quad (15)$$

- Find the equilibria of this system.
- Show that the $x_2$-axis is invariant and that the slope $dx_2/dx_1$ is infinite (vertical) on this line; find other lines in the plane on which the slope $dx_2/dx_1$ is infinite.
- Now seek isoclines on which $dx_2/dx_1 = c$ for finite $c$ (try $c = 0, .5, 1, 2$)
- Sketch these curves, and the associated slopes $dx_2/dx_1$ on top of these curves, in the $(x_1, x_2)$ plane.
- Conjecture the phase portrait from this information. (You may want to plug the system into MATLAB to see if your conjecture is correct.)

**References**


