Chapter 3

Feedback Control

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Control Lyapunov Functions for Asymptotic Stability:

Definition 3.1 (Full-State Feedback Control). Given \( \dot{x} = f(x, u), \ f(0, 0) = 0, \) with \( x \in \mathbb{R}^n, u \in \mathbb{R}^m. \) We seek a full-state feedback control \( \alpha: \mathbb{R}^n \to \mathbb{R}^m \) such that \( x_e = 0 \) is globally asymptotically stable for the closed-loop function \( \dot{x} = f(x, \alpha(x)). \)

Theorem 3.2 (Converse Lyapunov Theorem). If there exists a solution to the above problem, then there exists some continuously differentiable \( V: \mathbb{R}^n \to \mathbb{R} \) such that:

- \( V(0) = 0, V(x) > 0 \forall x \neq 0, \)
- \( V \) is radially unbounded,
- \( \dot{V}(x) < 0 \) for each \( x \neq 0. \)

Note that,

\[ \dot{V}(x) < 0 \text{ means } \forall x \neq 0, \dot{V}(x) = \frac{\partial V}{\partial x} f(x, \alpha(x)) < 0, \]

\[ \Rightarrow \forall x \neq 0, \exists u \in \mathbb{R}^m, \frac{\partial V}{\partial x} f(x, u) < 0, \text{ (take } u = \alpha(x) \text{ for instance)}, \]

\[ \Leftrightarrow \forall x \neq 0, \inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} (x) f(x, u) \right\} < 0. \]

Definition 3.3 (Artstein and Sontag: Control Lyapunov Function, CLF). A Control Lyapunov Function (CLF) is a continuously differentiable, positive definite, radially unbounded function \( V: \mathbb{R}^n \to \mathbb{R} \) satisfying \( \forall x \neq 0, \inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} (x) f(x, u) \right\} < 0. \)

Remark. This is the globally asymptotic stability version of the definition. There exist other versions, e.g. for asymptotic stability or exponential stability.
Theorem 3.4 (Artstein, 1983). Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a CLF for the system $\Sigma : \dot{x} = f(x,u), x \in \mathbb{R}^n, u \in \mathbb{R}^m$, $f$ Lipschitz continuous. Then there exists an infinitely continuously differentiable (i.e. smooth) feedback control $\alpha : \mathbb{R}^n \to \mathbb{R}^m$, such that the closed-loop system $\dot{x} = f(x, \alpha(x))$ is globally asymptotically stable.

Remark. The proof of this theorem is not constructive; i.e. it gives the existence of such a smooth feedback controller, but it does not tell us what this controller is. We will see theorems below that rectify this.

Definition 3.5 (Control Affine System). A control affine system is a system of the form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u$$

where $x \in \mathbb{R}^n, u_i \in \mathbb{R}, u \in \mathbb{R}^m, g_i : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, with:

$$g(x) = \begin{bmatrix} g_1(x) & \cdots & g_m(x) \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

Definition 3.6 (Lie Derivative). Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and let $\dot{x} = f(x)$. Then the Lie derivative of $h$ with respect to $f$ is defined as:

$$L_f h(x) \equiv \frac{\partial h}{\partial x} f(x).$$

Example. For a control affine system:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i,$$

a Lyapunov function $V(x)$ evolves as:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} \left[ f(x) + \sum_{i=1}^{m} g_i(x)u_i \right] = L_f V(x) + \sum_{i=1}^{m} L_{g_i}(x)u_i$$

$$= L_f V(x) + L_g V(x)u.$$

Proposition 3.7. The following statements are equivalent:

- For each $x \neq 0$, $\inf_{u \in \mathbb{R}^m} L_f V(x) + L_g V(x)u < 0$.

- For each $x \neq 0$, if $L_g V(x) = 0$, then $L_f V(x) < 0$. 
Proof. If $L_g V(x) = 0$, then the first statement is true if and only if, for each $x \neq 0$:

$$\inf_{u \in \mathbb{R}^m} L_f V(x) < 0$$

which in turn is true if and only if the second statement is true.

If $L_g V(x) \neq 0$, then, based on the values of $L_f V(x)$ and $L_g V(x)$, one can always choose $u$ such that $L_f V(x) + L_g V(x)u$ is as negative as possible, i.e.:

$$\inf_{u \in \mathbb{R}^m} L_f V(x) = -\infty < 0$$

\[ \blacksquare \]

Remark (Multi-input version).

$$\forall x \neq 0, \inf_{u \in \mathbb{R}^m} \{L_f V(x) + \sum_{i=1}^m L_g_i V(x)u_i\} < 0$$

$$\iff \forall x \neq 0, [L_{g_1} V(x) \ L_{g_2} V(x) \ \cdots \ L_{g_m} V(x)] = [0 \ 0 \ \cdots \ 0] \implies L_f V(x) < 0,$$

$$\iff \forall x \neq 0, \sum_{i=1}^m (L_g_i V(x))^2 = 0 \implies L_f V(x) < 0.$$ 

Example (min-norm Control). Consider a control affine system $\Sigma : \dot{x} = f(x) + g(x)u$, $u$ scalar. If $V(x)$ is a control Lyapunov function for $\Sigma$, then we could choose $u^*$ as follows:

$$u^* = \begin{cases} \frac{L_f V}{L_g V}, & L_f V > 0, \\ 0, & \text{else.} \end{cases}$$

i.e. $u^*$ solves the constrained optimization problem:

$$u^* = \text{Minimize } u^T u$$

subject to: $L_f V + L_g V \cdot u \leq 0$.

Remark. Note that the above minimization problem is a quadratic program since the cost is quadratic in $u$ and the constraints are linear in $u$.

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For multiple inputs $\{u_i | i = 1, \cdots, m\}$, the min-norm controller becomes:

$$u^* = \begin{cases} \frac{L_f V(x)}{L_g V(x)L_g V(x)^T L_g V(x)}L_g V(x)^T, & L_f V(x) > 0 \\ 0, & \text{else.} \end{cases}$$

Example. Consider the system $\Sigma : \dot{x} = x + x^2 u$, where $x, u \in \mathbb{R}$. If we want $x = 0$ to be asymptotically stable, then we need $x + x^2 u < 0$ when $x > 0$ (similar for $x < 0$). Look at $x + x^2 u = 0$ as being the limiting case, i.e.,

$$u = -\frac{x}{x^2} = -\frac{1}{x}.$$ 

In particular, for $x \to 0$, we have $|u| \to \infty$! In this case, we say that the system has the large control property.
As the above example illustrates, it is undesirable for a system to have the large control property. We give a definition for the opposite property below.

**Definition 3.8 (Small Control Property).** A control Lyapunov function satisfies the **small control property** if, for each $\epsilon > 0$, there exists some $\delta > 0$ such that for each $x \in B_\delta(0)$, there exists some $u(x) \in \mathbb{R}^m$ satisfying:

1. $\|u\| < \epsilon$

2. $\dot{V}(x, u) = L_f V(x) + L_g V(x)u < 0.$

**Theorem 3.9 (Sontag 1989, single input version).** Suppose $V$ is a CLF for the single-input system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ where $f(0) = 0$ and $f(\cdot), g(\cdot)$ are Lipschitz continuous. Then, choosing the control as

$$u = \alpha_s(x) := \begin{cases} \frac{-L_f V(x) + \sqrt{L_f V(x)^2 + L_g V(x)^2}}{L_g V(x)} & , \quad L_g V \neq 0, \\ 0, & \quad L_g V = 0, \end{cases}$$

results in:

1. $\dot{V}(x) = -\sqrt{L_f V(x)^2 + L_g V(x)^2} < 0$, for each $x \neq 0$, with $V(x) = 0$ if and only if $x = 0$. Moreover, $x = 0$ is globally asymptotically stable.

2. $\alpha_s(\cdot)$ is continuous for each $x \neq 0$.

3. $\alpha_s(\cdot)$ is continuous at $x = 0$ if $V(x)$ satisfies the small control property.

4. $\alpha_s(\cdot) \in C^k$ for each $x \neq 0$ if $V(\cdot) \in C^{k+1}$ and $f(\cdot), g(\cdot) \in C^k$.

**Proof.** (see Sontag’s 1989 paper)

**Example.** We wish to design controls for the system:

$$\Sigma: \quad \dot{x} = \sin x - x^3 + u, \quad x \in \mathbb{R}, u \in \mathbb{R},$$

i.e., $f(x) = \sin(x) - x^3, \quad g(x) = 1$. Consider the candidate control Lyapunov function: $V(x) = \frac{1}{2}x^2$.

First, let us check that $V(x)$ satisfies all the conditions in the definition of a control Lyapunov function. From its definition, $V \in C^1$, and $V$ is positive definite and radially unbounded. It remains to check whether, for each $x \neq 0$ such that $L_g V(x) = 0$, we have $L_f V(x) < 0$. Evaluating $V$, we have:

$$\dot{V} = \frac{\partial V}{\partial x} \dot{x} = x(\sin x - x^3 + u) = x \sin x - x^4 + x \cdot u,$$

$$\Rightarrow L_f V(x) = x \sin x - x^4, \quad L_g V(x) = x.$$

Since $L_g V(x) = x$ is zero if and only if $x = 0$, the final condition is automatically satisfied.

Consider the following controls applied to $\Sigma$ with the CLF choice $V(x) = \frac{1}{2}x^2$. 

1. $\alpha_s(x)$ (Sontag control)

From the definition of Sontag control, we have:

$$\alpha_s(x) = -\frac{x(\sin x - x^3)}{x} + \sqrt{x^2(\sin x - x^3)^2 + x^4}, \quad \forall x \neq 0$$

$$= -(\sin x - x^3) - \text{sgn}(x) \cdot \sqrt{(\sin x - x^3)^2 + x^2}, \quad \forall x \neq 0,$$

and $\alpha_s(0) = 0$. (In fact, in this particular case, $\alpha_s(x)$ is continuous at $x = 0$ and in $C^\infty$ at each $x \neq 0$.

In this case:

$$\dot{V}(x) = -\sqrt{x^2(\sin x - x^3)^2 + x^4} < 0, \quad \forall x \neq 0,$$

with equality if and only if $x = 0$.

2. $\alpha_1(x) = \sin x + x^3 - x$ (Feedback linearization)

We have, for the closed-loop system and $\dot{V}$:

$$\Sigma_{CL,1} : \dot{x} = -x,$$

$$\Rightarrow \dot{V}(x) = -x^2 \leq 0,$$

with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.

3. $\alpha_2(x) = \sin x - x$ (Only cancel destabilizing nonlinearities)

We have, for the closed-loop system and $\dot{V}$:

$$\Sigma_{CL,2} : \dot{x} = -x^3 - x,$$

$$\Rightarrow \dot{V}(x) = -x^4 - x^2 \leq 0,$$

with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.

4. $\alpha_3(x) = -\sin x$ (No need to add linear stabilizing term)

We have, for the closed-loop system and $\dot{V}$:

$$\Sigma_{CL,3} : \dot{x} = -x^3,$$

$$\Rightarrow \dot{V}(x) = -x^4 \leq 0,$$

with equality if and only if $x = 0$. Thus, the closed-loop system is globally asymptotically stable.
5. \( \alpha_4(x) = -x \) (Linear control that dominates destabilizing nonlinearities)

We have, for the closed-loop system and \( \dot{V} \):

\[
\Sigma_{\text{CL,4}} : \quad \dot{x} = \sin x - x^3 - x, \\
\Rightarrow \dot{V}(x) = x(\sin x - x) - x^4 \leq 0,
\]

with equality if and only if \( x = 0 \). Thus, the closed-loop system is globally asymptotically stable.

6. \( \alpha_{\min-\text{norm}} = \begin{cases} -\frac{L_f V}{L_g V}, & L_f V > 0, \\
0, & \text{else.} \end{cases} \)

Note that the min-norm controller stabilizes the system by definition.

**Remark.** Observe that, while the first five controllers render the resulting closed-loop system globally asymptotically stable and the last one renders the system stable, they do so in noticeably different ways. Whereas the norm of the Sontag control dies off when the magnitude of \( x \) becomes large, this is not true for \( \alpha_1(x), \alpha_2(x), \) or \( \alpha_4(x) \), all of which become unbounded as \( x \to \pm \infty \). This is mainly because of the inability of these three controllers to harness the intrinsic stabilizing capabilities of the \(-x^3\) term already present in the open-loop system (before the application of any control). See Figure 3.1 for a plot of the control effort for all the six controllers.

**CLF vs. Linearization:**

Consider the following example, which contrasts the use of Control Lyapunov Functions (CLFs) with the use of linearization in solving control problems.

**Example.** Consider a model of a pendulum without the cart:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{-x_2^2 \sin 2x_1 + 4 \sin x_1}{4 + 2 \sin^2 x_1} + \frac{-2 \cos x_1}{2 + \sin^2 x_1} u.
\end{align*}
\]

Linearizing the system about the origin, we have:

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u = A x + B u.
\]
Figure 3.1: Comparing various controllers for stabilizing the system \( \dot{x} = \sin x - x^3 + u \) Note that the Sontag and min-norm controller have control effort that goes to zero as the state \( x \) increases.

We can use state feedback, i.e. a control of the form \( u = Kx \), where \( K \in \mathbb{R}^{1 \times 2} \), to place the eigenvalues at \((-2, -3)\). The closed-loop linearized system then becomes:

\[
\dot{x} = (A + BK)x = \overline{A}x,
\]

where:

\[
\overline{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}.
\]

To find a suitable Lyapunov function, we must solve the Lyapunov equation \( \overline{A}^T P + P \overline{A} = -Q \), where \( Q \) is positive semidefinite (i.e. \( Q \geq 0 \)). In particular, below, we will choose \( Q = I \). We can then solve the above Lyapunov equation to obtain:

\[
P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix}
\]

Diagonalizing \( P \), we find that \( P = V D V^T \), where:

\[
V = \begin{bmatrix} -0.9966 & 0.0824 \\ -0.0824 & -0.9966 \end{bmatrix}, \quad D = \begin{bmatrix} 1.1236 & 0 \\ 0 & 0.1098 \end{bmatrix}.
\]

To estimate the region of attraction, we want to determine the largest \( c > 0 \) such that:

\[
V^{-1}\{(\infty, c)\} \subset \dot{V}^{-1}\{(\infty, 0)\} \bigcup \{0\},
\]
where $\dot{V}$ is evaluated in the closed-loop system, using $u = Kx$:

$$
\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) \cdot u \\
= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) \cdot kx.
$$

Numerically, the region of attraction can be approximated as shown below.

Figure 3.2: Zero Level Set for $\dot{V}$. The region in which $\dot{V} > 0$ is labeled with small " + " symbols; the region in which $\dot{V} < 0$ is labeled with small " o " symbols.
Below, we will attempt to enlarge the region of attraction estimated above by using a non-linear controller rather than a linear one. Our simplest candidate for the Lyapunov function is the following quadratic $V(x)$:

$$V(x) = \frac{1}{2} x^T P x,$$

where:

$$P = \begin{bmatrix} 1.1167 & 0.0833 \\ 0.0833 & 0.1167 \end{bmatrix}.$$

Then, we have:

$$\frac{\partial V}{\partial x} \cdot f(x) = (1.1167x_1 + 0.0833x_2)x_2 + (0.0833x_1 + 0.1167x_2)f_2(x_1, x_2),$$

$$\frac{\partial V}{\partial x} \cdot g(x) = (0.0833x_1 + 0.1167x_2)g(x_1, x_2).$$

The region in which $V(x)$ is a CLF is given by the region in which, whenever $\frac{\partial V}{\partial x} \cdot f(x) = 0$, we have $\frac{\partial V}{\partial x} \cdot g(x) < 0$. Observe that $\frac{\partial V}{\partial x} \cdot g(x) = 0$ if and only if:

$$x_2 = -\frac{1}{2} x_1,$$

or

$$x_1 = \pm \left( k + \frac{1}{2} \right) \pi, \quad k \in \mathbb{N} \cup \{0\}.$$
To verify that $V(x)$ is indeed a suitable candidate for a control Lyapunov equation, we must check whether, for each nonzero $x$ in the set described above, we have $\frac{\partial V}{\partial x} f(x) < 0$:

$$\left. \frac{\partial V}{\partial x} f(x) \right|_{x_2 = -0.7143x_1} = \left[ 1.1167x_1 + 0.0833(-0.7143x_1) \right] \cdot (-0.7143x_1)$$

$$+ \left[ 0.0833x_1 - 0.1167(-0.7143x_1) \right] \cdot \frac{(-0.7143x_1)^2 \sin 2x_1 + 4 \sin x_1}{4 + 2 \sin^2 x_1}$$

This quantity, is indeed non-positive for each $x_1 \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$, as shown graphically below. We thus conclude that there exists some $u = \alpha(x)$ such that $\dot{V} = L_f V + L_g V \cdot \alpha < 0$ for each $x_1 \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \setminus \{0\}$.

![Figure 3.4: $L_f V$ in a region where $L_g V = 0$](image)

The next figure shows that the region of convergence estimated by the non-linear controller displays performance exceeding that of the linear controller.
Figure 3.5: Linear vs. Nonlinear ROC. It can be demonstrated that the nonlinear controller provides a region of attraction $R_A \supset \{ x | V(x) \leq 2.3 \}$, which is more than twice the size of the region of attraction $R_A \supset \{ x | V(x) \leq 0.8 \}$ predicted by the linear controller.

Figure 3.6: Sontag Feedback Controller
The Sontag controller (formula reiterated below) provides a nonlinear controller with a very complicated form. However, it could be implemented using look-up tables or using a lower-polynomial approximation. An illustration is provided above.

Finally, we provide a figure for one additional nonlinear controller of the form:

$$u = \frac{7x_1 + 5x_2}{\cos x_1}$$

as shown below.

![Figure 3.7: Alternate Nonlinear Feedback Control](image)

In summary, the key points of this handout were as follows:

1. Sontag’s formula is a useful starting point.

2. Sontag’s formula provides only one of an infinite number of positive feedback control strategies that impose $\dot{V} < 0$.

3. Physical insight on real problems are useful in finding other feedback controls.

4. When necessary, do numerical computations.