1) a). \[ \dot{x}_1 = -x_1^3, \]
\[ \dot{x}_2 = x_1. \]

(0,0) is not SISL.

Proof. Choose \( \varepsilon = 1. \)

If \( x_1(0) > 0, \) then \( \dot{x}_2 > 0 \) always.

So \( \| x(t) \| > \varepsilon \) in finite time,

so \( \not\exists \delta > 0 \) s.t.

\( \| x(0) \| < \delta \Rightarrow \| x(t) \| < \varepsilon \ \forall t > 0. \)

\[ \therefore \] Not SISL. \( \Box \)
\[ x = y - f(x) \]
\[ y = -x \]

Consider the stored energy:
\[ W = \frac{1}{2} (x^2 + y^2) \]
\[ \frac{dw}{dt} = xx + yy = x(y - f(x)) + y(-x) \]
\[ = -x f(x) \leq 0 \]
\[ \therefore \frac{dw}{dt} \begin{cases} < 0 & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0. \end{cases} \]

Now the equilibrium is at \( x = 0 \)
\[ y = f(x) \]
\[ = f(0). \]

Show it is stable:
1. if \( x \neq 0 \) \( \frac{dw}{dt} < 0 \)
2. if \( x = 0, y \neq 0 \)
\[ f(0) = 0 \]
\[ x = y = f(0) \neq 0 \]
\[ \Rightarrow x \text{ become non-zero, then (1) applied.} \]

\[ \therefore \text{The equations admit only one stable equilibrium at } (x, y) = (0, 0). \]
Problem 3

**Theorem statement:**

If $V(x, t)$ is locally positive definite and decrescent around an equilibrium point $x = 0$, and $-\dot{V}(x, t)$ is locally positive definite around $x = 0$, then $x = 0$ is asymptotically stable.

**Proof (Version Mo; alternatively, see Sastry 1999 pages 189-191):**

Since $V(x, t)$ is decrescent, we have $V(x, t) \leq \beta(|x|)$ for some class $K$ function $\beta(\cdot)$. Also, since $-\dot{V}(x, t)$ is locally positive definite, $V(x(t), t)$ is decreasing, so there exists $c$ such that the set $\Omega_c = \{x | \beta(|x|) = c\}$ is positively invariant.

Consider the set $B_\epsilon = \{x | \|x\| \leq \epsilon\}$ that contains $\Omega_c$, and the set $B_\delta = \{x | \|x\| \leq \delta\}$ that is contained in $\Omega_c$; note $B_\delta \subseteq \Omega_c \subseteq B_\epsilon$. Then, any trajectory starting in $B_\delta$ will remain in $\Omega_c$, and thus remain in $B_\epsilon$ since we have $\Omega_c$ is positively invariant. Therefore, $x = 0$ is stable in the sense of Lyapunov.

Next, since $V(x(t), t)$ is locally positive definite, it is bounded below by 0. Because $V(x(t), t)$ is also decreasing, $V(x(t), t)$ must converge to some value $v \geq 0$. We now show that $v = 0$ by contradiction.

Suppose $v > 0$, then let $-\gamma = \max_{x \in B_\delta \setminus \{x | V(x, t) \geq v\}} \dot{V}(x, t)$. Since $-\dot{V}(x, t)$ is locally positive definite, we must have that $-\gamma < 0$. From this, we get $\dot{V}(x, t) \leq -\gamma \Rightarrow V(x, t) \leq V(x_0, t) - \gamma t \forall x_0 \in B_\delta$.

This implies that $V(x, t) < 0$ when $t > \frac{V(x_0, t)}{-\gamma}$, a contradiction since $V(x, t)$ is locally positive definite. Therefore, $v = 0$, and we have $V(x, t) \to 0 \Rightarrow x(t) \to 0.$
Problem 4

\[ \dot{x}_1 = -x_1^3 + u \quad u = -x_2 \]
\[ \dot{x}_2 = x_1 \]

\[ \Rightarrow \text{try } V(x) = x_1^2 + x_2^2 \]

\[ \dot{V}(x) = 2x_1(-x_1^3 - x_2) + 2x_2x_1 \]
\[ = -2x_1^4 \]

\[ \Rightarrow \dot{V} \leq 0 \]

\[ \Rightarrow (0, 0) \text{ is SISL} \]

Using LaSalle, consider

\[ S = \{ x | V(x) = 0 \} \]
\[ = \{ x | x_1 = 0 \} \]

For \( x_1 = 0 \) \( \forall t \geq 0 \), \( \dot{x}_1 = 0 \), which means that \(-x_1^3 - x_2 = 0\), but \( x_1 = 0 \)

\[ \Rightarrow x_2 = 0 \]

\[ \therefore \text{the largest invariant set in } S \text{ is} \]
\[ M = \{ x | x_1 = 0, x_2 = 0 \} \]

By LaSalle's Thm, \( (0, 0) \) is globally asymptotically stable.
\[ y''(t) + (a + b \cos y(t)) y' + c \sin y(t) = 0 \]
\[ x_1 = y, \quad x_2 = y' \]
\[ x_1' = x_2 \]
\[ x_2' = -(a + b \cos x_2(t)) x_2 + c \sin x_1(t) \]

(a) \((0,0)\) is an equilibrium. 
\[ V(x) = \frac{x_2^2}{2} + c(1 - \cos x_1) \]
\[ V(0,0) = 0 \]

\[ 0 \leq 1 - \cos x_1 \leq 2 \]

if we restrict \( x_1 \in (-\pi, \pi) \) to ensure 
\[ 1 - \cos x_1 = 0 \] only at \( x_1 = 0 \) and that as 
\[ |x_1| \to \infty, \quad (1 - \cos x_1) \to \]

Assume \( c > 0 \)

\[ \therefore \text{on } x_1 \in (-\pi, \pi), \]
\[ V(x_1, x_2) \geq c x \frac{x_1^2}{2} + \frac{x_2^2}{2} \text{ where } x > 0 \text{ is chosen appropriately to make } \]
\[ \frac{x_1^2}{2} < 1 - \cos x_1 \]

\[ \therefore V(x_1, x_2) > x'\|x_1| \]

\[ \therefore V \text{ is an lpdf} \]

\[ \therefore \dot{V}(x_1, x_2) = c \sin x_1 x_1' + x_2 x_2' \]
\[ = c \sin x_1 \cdot x_2 - x_2^2 \left( a + b \cos x_1 \right) + c x_2 \sin x_1 (t) \]
\[ = -x_2^2 (a + b \cos x_1) + c \sin x_1 (t) \]
\[ \therefore -\dot{V}(x_1, x_2) \geq 0 \implies (0,c) \text{ is stable} \]
(b). To show $(0,0)$ is an asymptotically stable equilibrium point, we need to show that $V$ is an lpdf, decreasing and $-\dot{V}$ is an lpdf, $a > b > 0$

$V(x_1, x_2)$ is an lpdf on $x_1 \in (-\pi, \pi)$ by (a).

$-\dot{V} = -x_2^2 \ (a + b \cos x_1)$ where $a > b > 0$

so at $(0,0)$, $-\dot{V} = 0$, use Lasalle's Thm:

$S_c = \{ x \in \mathbb{R}^2 : V(x) \leq c \}^3$

so $S_c$ is bounded and $\dot{V} \leq 0 \ \forall x \in S_c$ (from (a)).

Define $S \subset S_c$ by $S = \{ x \in S_c : \dot{V}(x) = 0 \}^3$

$\therefore S = \{ x \in S_c : a + b \cos x_1 = 0 \ \text{or} \ x_2 = 0 \}^3$

Let us restrict $S_c$ such that $\forall x \in S_c \ x_1 \in (-\pi, \pi)$

to ensure that $a + b \cos x_1 > 0 \ \forall x \in S_c$

(this restriction is without loss of generality, since it can be translated to a restriction on the constant $c$).

$\therefore S = \{ x \in S_c : x_2 = 0 \}^3$

but $x_2 = 0 \Rightarrow \dot{x_2} = 0 \Rightarrow \sin x_1 = 0$

$\Rightarrow x_1 = 0$

Letting $M$ be the largest invariant set in $S$, we have $M = \{ (0,0) \}^3$, and by Lasalle, the origin is asymptotically stable.
Problem 6

\[ I \dot{w} + w \times I w = -cw \dot{d}_0 + d_0 \times b_0 \]
\[ \dot{d}_0 = -w \times d_0. \]

To find the equilibria, set \( \dot{w} = 0 \), \( \dot{d}_0 = 0 \)

\[ \therefore \quad w \times I w = -cw \;
\text{and} \quad -w \times d_0 = 0 \implies \dot{d}_0 = k_1 w \text{ where } k_1 \text{ is constant.} \]

\[ \therefore (w \times I w) - k_1 (w \times b_0) = -cw \quad \text{(eqn \(*\))} \]

\[ \begin{align*}
\text{now} \quad (w \times I w) \cdot w &= 0, \\
(w \times b_0) \cdot w &= 0 \\
\implies \left[ (w \times I w) - k_1 (w \times b_0) \right] \cdot w &= 0
\end{align*} \]

\[ \therefore \text{The left hand side of (*) is normal (perpendicular) to } w, \text{ and the right hand side is a linear multiple of } w. \]

Solving (*) for \( w \), we have \( w = 0 \)

\[ \therefore \quad d_0 \times b_0 = 0 \]
\[ \therefore \quad d_0 = k_2 b_0 \text{ where } k_2 \text{ is constant.} \]

now \( d_0 \) and \( b_0 \) are unit vectors, so \( k_2 = \pm 1 \).

\[ \therefore \quad d_0 = \pm b_0. \]

\[ \therefore \text{The two equilibria are} \]
\[ \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \text{ and } \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} -b_0 \end{bmatrix} \]
We now use Lyapunov theory to determine the stability of the equilibria:

\[ V(w, d_0) = \frac{1}{2} w^T I w + \frac{1}{2} \| d_0 + b_0 \|^2 \]

is a candidate Lyapunov function.

\[ V(0, b_0) = 2, \quad V(0, -b_0) = 0 \]

Since \( I > 0 \), \( V(w, d_0) \) is positive definite.

Also,

\[ \dot{V}(w, d_0) = \frac{1}{2} \dot{w}^T I w + \frac{1}{2} w^T \dot{I} w + \frac{1}{2} (\dot{d}_0 + \dot{b}_0)^T (d_0 + b_0) + \frac{1}{2} (d_0 + b_0)^T (\dot{d}_0 + \dot{b}_0) \]

Now each of the terms above is a scalar, and \( \dot{d}_0 = 0 \), so

\[ \dot{V}(w, d_0) = \dot{w}^T I w + \dot{d}_0^T (d_0 + b_0) \]

\[ = (-\alpha w + d_0 x b_0 - w x I w)^T w + (-w x d_0)^T (d_0 + b_0) \]

\[ = -\alpha w^T w + (d_0 x b_0)^T w - (w x I w)^T w - (w x d_0)^T d_0 - (w x d_0)^T b_0 \]

\[ < 0 \]
now \( F \cdot G \times H = G \cdot H \times F = H \cdot F \times G \).
\[
\vdash (d_0 \times b_0)^T \ w = - (w \times d_0)^T \ b_0
\]
\[
\vdash \dot{v}(w, d_0) = - \alpha \ |w|^2
\]
\[
\vdash - \dot{v}(w, d_0) > 0 \Rightarrow \text{system is Stable in the sense of Lyapunov.}
\]

Applying Casali's Theorem:

Looking first at eq. point \((0, -b_0)\):
Choose \(c_1 < 2\) and define

\[
\mathcal{R}_{c_1} := \{ (w, d_0) \in \mathbb{R}^b, \ |d_0| = 1 | \dot{v}(w, d_0) \leq c_1 \}
\]

So \((0, b_0) \not\in \mathcal{R}_{c_1}\).

Define \( S = \mathcal{R}_{c_1} \) as

\[
S := \mathcal{R}_{c_1} \quad \dot{v}(w, d_0) = 0
\]

\[
= \mathcal{R}_{c_1} \quad \dot{v}(w, d_0) = 0
\]

Let \( M \) be the largest invariant set in \( S \):

\[
M := \{ (w, d_0) \in S | \ |w| = 0 \ and \ \dot{w} = 0 \}
\]

\((\dot{w} = 0 \Rightarrow w = 0 \Rightarrow d_0 = 0 \Rightarrow M = \{ (0, -b_0) \})\)

\[
\vdash \text{By Casali's Theorem, locally}
\]

\((0, -b_0) \text{ is asymptotically stable.}
\]

\[
\vdash \text{all initial conditions in } \mathcal{R}_{c_1} \text{ converge to } (0, -b_0).
\]