1 Linear and Time-Invariant Dynamical Systems

Definition 1. \((U, Y, \Sigma, s, r)\) is said to be a \textit{linear dynamical system} if

- \(U, \Sigma, Y\) are vector spaces over the same field;
- \(\rho\) is linear in both \(x_0\) and \(u_0\), i.e.,
\[
\rho(t_1, t_0, \alpha_1 x_{01} + \alpha_2 x_{02}, \alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \rho(t_1, t_0, x_{01}, u_1) + \alpha_2 \rho(t_1, t_0, x_{02}, u_2).
\]

Definition 2. \((U, Y, \Sigma, s, r)\) is said to be a \textit{time-invariant dynamical system} if

- We define a shift operator \(T_\tau : F \to F\) for \(F = U\) or \(Y\), such that
\[
(T_\tau(f))(t) = f(t - \tau)
\]
- \(U\) and \(Y\) are closed under \(T_\tau\) for all \(\tau\).
- For all \(t_0, t_1 \geq t_0, \tau \in \mathcal{T}\), for all \(x_0 \in \Sigma\), for all \(u \in U\)
\[
\rho(t_1, t_0, x_0, u) = \rho(t_1 + \tau, t_0 + \tau, x_0, T_\tau(u))
\]

Differential equation representation of a linear system

\[
\begin{align*}
\dot{x}(t) &= ax(t) + bu(t) \\
y(t) &= cx(t) + du(t) \\
x(t_0) &= x_0,
\end{align*}
\]

where \(x(t) \in \mathbb{R}, u(t) \in \mathbb{R}, y(t) \in \mathbb{R},\) and \(a, b, c, d \in \mathbb{R}\).

Problem 1. The solution to the differential equation in \(x\) given by (3) is

\[
x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^{t} e^{a(t-\tau)}bu(\tau)d\tau
\]

Is this system linear? Time-invariant?
2 The State-Transition Matrix $\Phi$

**Definition 3.** The matrix-valued function $\Phi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is called the state-transition matrix of $A$ if $\Phi(\cdot, t_0)$ solves the matrix differential equation

\[
\dot{X}(t) = A(t)X(t), \quad X(t) \in \mathbb{R}^{n \times n}
\]

\[X(t_0) = I.\]

Consider the system described by the vector differential equation

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

\[x(t_0) = x_0 \quad (3)\]

**Proposition 4.** The solution to (3) is given by

\[x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t}\Phi(t, \tau)B(\tau)u(\tau)d\tau.\]

**Proposition 5.** For all $t, t_0, t_1$ in $\mathbb{R}_+$

\[\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0) \quad (4)\]

**Problem 2.** Show that $\Phi(t, t_0)$ is invertible and that $(\Phi(t, t_0))^{-1} = \Phi(t_0, t)$.

**Problem 3** (From Fall 2009 Midterm). For a nonsingular $M(t) \in \mathbb{R}^{n \times n}$, determine an expression for

\[
\frac{d}{dt}M^{-1}(t)
\]

in terms of $\dot{M}(t)$ and $M^{-1}(t)$.

**Problem 4** (From Fall 2014 Midterm). Find an expression for

\[
\frac{d}{d\tau}\Phi(t, \tau).
\]
Problem 5. Prove that $\Phi(t_0, t)$ is the unique solution to the matrix differential equation:

$$\frac{d}{dt}X(t) = -X(t)A(t), \ X(t_0) = I.$$

Hint: Use the result obtained in Problem 4.

Problem 6. Calculate the state transition matrix for $\dot{x}(t) = A(t)x(t)$ for the following $A(t)$:

$$A(t) = \begin{bmatrix} t & 2 \\ 0 & -1 \end{bmatrix}$$