Problem 1: Perturbed nonlinear systems.
Suppose that some physical system obeys the differential equation
\[ \dot{x} = p(x, t), \quad x(t_0) = x_0, \quad \forall t \geq t_0 \]
where \( p(\cdot, \cdot) \) obeys the conditions of the fundamental theorem. Suppose that as a result of some perturbation the equation becomes
\[ \dot{z} = p(z, t) + f(t), \quad z(t_0) = x_0 + \delta x_0, \quad \forall t \geq t_0 \]
Given that for \( t \in [t_0, t_0 + T] \), \( ||f(t)|| \leq \epsilon_1 \) and \( ||\delta x_0|| \leq \epsilon_0 \), find a bound on \( ||x(t) - z(t)|| \) valid on \( [t_0, t_0 + T] \).

Problem 2: Dynamical systems, time invariance.
Suppose that the output of a system is represented by
\[ y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} u(\tau) d\tau \]
Is the system time invariant? You may select the input space \( U \) to be the set of bounded, piecewise continuous, real-valued functions defined on \((-\infty, \infty)\).

Problem 3: Solution of a matrix differential equation.
Let \( A_1(\cdot), A_2(\cdot), \) and \( F(\cdot) \), be known piecewise continuous \( n \times n \) matrices. Let \( \Phi_i \) be the transition matrix of \( \dot{x} = A_i(t)x \), for \( i = 1, 2 \). Show that the solution of the matrix differential equation:
\[ \dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t), \quad X(t_0) = X_0 \]
is
\[ X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)F(\tau)\Phi_2^T(t, \tau) d\tau \]

Problem 4: Satellite Problem, linearization, state space model.
Model the earth and a satellite as particles. The normalized equations of motion, in an earth-fixed inertial frame, simplified to 2 dimensions (from Lagrange’s equations of motion, the Lagrangian \( L = T - V = \frac{1}{2}r^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{k}{r} \)):
\[ \ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \]
\[ \ddot{\theta} = -2\frac{\dot{r}}{r} + \frac{1}{r}u_2 \]
with \( u_1, u_2 \) representing the radial and tangential forces due to thrusters. The reference orbit with \( u_1 = u_2 = 0 \) is circular with \( r(t) \equiv p \) and \( \theta(t) = \omega t \). From the first equation it follows that \( p^3 \omega^2 = k \). Obtain the linearized equation about this orbit.

**Problem 5: State Transition Matrix, calculations.**

Calculate the state transition matrix for \( \dot{x}(t) = A(t)x(t) \), with the following \( A(t) \):

(a) \( A(t) = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} \); (b) \( A(t) = \begin{bmatrix} -2t & 0 \\ 1 & -1 \end{bmatrix} \); (c) \( A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix} \)

Hint: for part (c) above, let \( \Omega(t) = \int_0^t \omega(t')dt' \); and consider the matrix

\[
\begin{bmatrix}
\cos \Omega(t) & \sin \Omega(t) \\
-\sin \Omega(t) & \cos \Omega(t)
\end{bmatrix}
\]

(d) For both of systems (a) and (b) above, describe the zero input (non-zero initial state) response.

**Problem 6: Sampled Data System**

You are given a linear, time-invariant system

\[ \dot{x} = Ax + Bu \] (1)

which is sampled every \( T \) seconds. Denote \( x(kT) \) by \( x(k) \). Further, the input \( u \) is held constant between \( kT \) and \( (k+1)T \), that is, \( u(t) = u(k) \) for \( t \in [kT, (k+1)T] \). Derive the exact state equation for the sampled data system, that is, give a formula for \( x(k+1) \) in terms of \( x(k) \) and \( u(k) \).