**Problem 1.** Consider an inner product space $V$, with $x, y \in V$. Show, using properties of the inner product, that
\[ ||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2 \]
where $|| \cdot ||$ is the norm induced by the inner product.

**Problem 2.** Consider an inner product space $(\mathbb{C}^n, \mathbb{C})$, equipped with the standard inner product in $\mathbb{C}^n$, and a map $A : \mathbb{C}^n \to \mathbb{C}^n$ which consists of matrix multiplication by an $n \times n$ matrix $A$. Find the adjoint of $A$.

**Problem 3:** Adjoints. Suppose that $A : V \to W$ is a linear map, and $V$ and $W$ are two inner product spaces. Prove that the adjoint map $A^* : W \to V$ is linear.

**Problem 4: Adjoints.** Consider a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, defined as:
\[
A \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
0 \\
x_1 \\
\vdots \\
x_{n-1}
\end{bmatrix}
\]
What is $A^*$?

**Problem 5:** Singular values. You are given a matrix $A \in \mathbb{R}^{n \times n}$, and you are told it has singular values $\sigma_1 > \sigma_2 > \ldots > \sigma_n > 0$ but you are not told what these singular values are. You wish to develop a simple iterative calculation to estimate the largest singular value $\sigma_1$.

Consider the following iterative calculation:

- Choose any $x_0 \in \mathbb{R}^n$
- Calculate $x_{i+1} = A^*Ax_i$, $i = 0, 1, 2 \ldots$
- Consider the sequence:
\[
\frac{||x_1||^2}{||x_0||^2}, \frac{||x_2||^2}{||x_1||^2}, \frac{||x_3||^2}{||x_2||^2}, \ldots
\]

The claim is that the sequence converges to $\sigma_1^2$.

(a) Determine whether or not this is true.

(b) If it is generally true, are there some $x_0$ for which the iterative calculation does not work?

**Problem 6: Local or global Lipschitz condition.** Consider the pendulum equation with friction and constant input torque:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 + \frac{T}{ml^2}
\end{align*}
\]
where $x_1$ is the angle that the pendulum makes with the vertical, $x_2$ is the angular rate of change, $m$ is the mass of the bob, $l$ is the length of the pendulum, $k$ is the friction coefficient, and $T$ is a constant torque. Let $B_r = \{ x \in \mathbb{R}^2 : ||x|| < r \}$. For this system (represented as $\dot{x} = f(x)$) find whether $f$ is locally Lipschitz in $x$ on $B_r$ for sufficiently small $r$, locally Lipschitz in $x$ on $B_r$ for any finite $r$, or globally Lipschitz in $x$ (ie. Lipschitz for all $x \in \mathbb{R}^2$).

**Problem 7: Existence and uniqueness of solutions to differential equations.**

Consider the following two systems of differential equations:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + e^t \cos(x_1 - x_2) \\
\dot{x}_2 &= -x_2 + 15 \sin(x_1 - x_2)
\end{align*}
\]

and

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1 x_2 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

(a) Do they satisfy a global Lipschitz condition?

(b) For the second system, your friend asserts that the solutions are uniquely defined for all possible initial conditions and they all tend to zero for all initial conditions. Do you agree or disagree?