EE21A Problem Set 5 – Solutions

Problem 1. Note that from the read-out map we have, \( y_T^T Q y_T = x_T^T C^T Q C x_T \). Thus the original LQR problem can be re-written as:

\[
\min_{x,u} \sum_{\tau=0}^{N-1} (x_T^T \bar{Q} x_T + u_T^T R u_T),
\]

which is the standard LQR problem with \( \bar{Q} \) as the state penalty matrix and \( Q_f = 0 \). The optimal cost-to-go and the optimal control at time \( t \) are thus given by:

\[
J_t^*(z) = z^T P_t z, \quad u_t^* = -K_t z,
\]

where \( t \in \{0, 1, \ldots, N - 1\} \) and

\[
P_t = \bar{Q} + K_t^T R K_t + (A - B K_t)^T P_{t+1} (A - B K_t), \quad P_N = 0 \quad (2)
\]

\[
K_t = (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A. \quad (3)
\]

Problem 2. We can code up the optimal control policy derived in Problem-1 to analyze the system. Here are the plots for the optimal control, output and cost-to-go for the three cases:

![Figure 1: Used control authority](image-url)
Since the output penalty is higher in case (ii), output is quickly driven to zero compared to the other two cases (see Figure 2). To do so, a higher control authority is used, as evident from Figure 1. On the other hand, when input penalty is higher, the control becomes very expensive. Thus, used control magnitude is very small as evident from Figure 1. As a result, output is not driven to zero even by the end of the horizon.

Problem 3. define time-varying constant $c_t$ that captures the deviation between the feasible trajectory dynamics and the actual next state:

$$c_t = f(x_t^*, u_t) - x_{t+1}^*$$

$$x_{t+1}^* = f(x_t^*, u_t) - c_t$$
Define new state: $z_t = x_t - x_t^*$, and new control $v_t = u_t - u_t^*$ therefore:

$$z_{t+1} = x_{t+1} - x_{t+1}^* = x_{t+1} - (f(x_t^*, u_t^*) - c_t)$$

$$= Ax_t + Bu_t - Ax_t^* - Bu_t^* + c_t$$

$$= A(x_t - x_t^*) + B(u_t - u_t^*) + c_t$$

$$= A(z_t) + B(v_t) + c_t$$

Now let’s augment our state variable to handle the constant term. Define new state: $\gamma_t = [z_t, 1]^{\top}$

$$\gamma_{t+1} = \begin{bmatrix} z_{t+1} \\ 1 \end{bmatrix} = \tilde{A} \begin{bmatrix} z_t \\ 1 \end{bmatrix} + \tilde{B}v_t$$

where $\tilde{A} = \begin{bmatrix} A & c_t \\ 0 & 1 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$. Use these new matrices to find the cost, optimal control.

**Problem 4.** (a) The optimal control of the infinite horizon problem is given by

$$u(t) = -\frac{1}{\rho}B^\top P x(t),$$

where $P$ is the unique positive definite matrix that solves the continuous-time algebraic Riccati equation

$$A^\top P + PA - \frac{1}{\rho}PBB^\top P + C^\top C = 0.$$ 

Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. We can rewrite the Riccati matrix equation as the following set of scalar equations:

$$1 - \frac{1}{\rho}p_2^2 = 0$$

$$2p_2 - \frac{1}{\rho}p_3^2 = 0$$

$$p_1 - \frac{1}{\rho}p_2p_3 = 0$$

Solving this set of equations, we have $p_1 = \sqrt{2}\rho^{1/4}$, $p_2 = \sqrt{\rho}$, and $p_3 = \sqrt{2}\rho^{3/4}$. Therefore, the optimal control is

$$u(t) = -\rho^{-1/2}x_1(t) - \sqrt{2}\rho^{-1/4}x_2(t)$$

(b) The closed-loop system is given by

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ -\rho^{-1/2} & \sqrt{2}\rho^{-1/4} \end{bmatrix} x =: \tilde{A}x.$$ 

The eigenvalues of $\tilde{A}$ are $\lambda = 2^{-1/2}\rho^{-1/4}(-1 \pm j)$. The first thing to note is that the system is stable since the eigenvalues are in the left half plane. Moreover, we see that for increasing $\rho$ the eigenvalues approach the imaginary axis (in fact, the origin) and the system response becomes slower. Conversely,
as \( \rho \to 0 \), the eigenvalues move far left in the complex plane and the system response becomes faster. This is intuitive, as \( \rho \) is the control weighting matrix, so a low \( \rho \) corresponds to a “cheap” control. In this case the state cost dominates, and the optimal controller tries to drive the state to zero fast using a high gain.