Problem 1. Suppose that \( \dim N(A) = k \). If \( k = n \), then \( R(A) = \{ \theta \} \), which has dimension zero, and the expression holds. Let \( \{ u_1, \ldots, u_k \} \) be a basis for \( N(A) \). We may extend \( \{ u_1, \ldots, u_k \} \) to a basis \( \{ u_1, \ldots, u_n \} \) for \( U \). We claim that \( \{ A(u_{k+1}), \ldots, A(u_n) \} \) is a basis for \( R(A) \). Observe first that \( R(A) = \text{span} \{ A(u_1), \ldots, A(u_n) \} = \text{span} \{ A(u_{k+1}), \ldots, A(u_n) \} \). Moreover, suppose that \( \sum_{i=k+1}^{n} b_i A(u_i) = 0 \) for some \( b_{k+1}, \ldots, b_n \in \mathbb{F} \). Then \( A(\sum_{i=k+1}^{n} b_i u_i) = 0 \). Hence, \( \sum_{i=k+1}^{n} b_i u_i \in N(A) \). Therefore, there exist \( c_1, \ldots, c_k \) such that \( \sum_{i=k+1}^{n} b_i u_i = \sum_{i=1}^{k} c_i u_i \). However, since \( \{ u_1, \ldots, u_n \} \) is linearly independent, we must have \( b_{k+1} = \ldots = b_n = 0 \). Thus we conclude that the set \( \{ A(u_{k+1}), \ldots, A(u_n) \} \) is linearly independent. Thus we have shown that \( \{ A(u_{k+1}), \ldots, A(u_n) \} \) is a basis for \( R(A) \). Hence, \( \dim R(A) = n - k = \dim(U) - \dim N(A) \), which implies that \( \dim R(A) + \dim N(A) = n \).

Problem 2.
If \( (x - x_0) \in N(A) \rightarrow A(x) = b \)

- \( A(x - x_0) = \theta_V \)
- \( A(x) - A(x_0) = \theta_V \)
- \( A(x) - b = \theta_V \)
- \( A(x) = b \)

If \( A(x) = b \rightarrow (x - x_0) \in N(A) \)

- \( A(x) - A(x_0) = b - b \)
- \( A(x) - A(x_0) = \theta_V \)
- \( A(x - x_0) = \theta_V \)
- \( (x - x_0) \in N(A) \)

Problem 3. With respect to the standard bases:

\[
A = \begin{bmatrix}
1 & 3 \\
2 & 5 \\
7 & 9
\end{bmatrix}
\]

From Discussion Section 2, we have that \( \bar{A} = QAP \), where \( Q = \bar{S}^{-1}S \) and \( P = R^{-1} \bar{R} \) with

\[
S = I_{3 \times 3}, \quad \bar{S} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}, \quad R = I_{2 \times 2}, \quad \bar{R} = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]

Thus we find

\[
\bar{A} = QAP = \bar{S}^{-1}SAR^{-1} \bar{R} = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 3 \\
2 & 5 \\
7 & 9
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} = \begin{bmatrix}
4 & 3 \\
-5 & -1 \\
12 & 6
\end{bmatrix}
\]
Problem 4. From the rank-nullity theorem, we have \( \dim \ker(A) = n - k \). Let \( \{v_i\}_{i=1}^n \) be a basis of \( \ker(A) \). Then \( A(v_i) = 0 \) for \( i = k + 1, \ldots, n \). Since the zero vector has all its coordinates zero in any basis, the last \( n - k \) columns of \( A \) are zero. It remains to be shown that we can complete the basis for \( U \) and choose a basis for \( V \) such that the first \( k \) columns are as desired. The form of the matrix \( A \) tells us that we want the \( i \)-th basis vector of \( V \) to be \( A(v_i) \) for \( i = 1, \ldots, k \). Let the basis for \( U \) be \( B_U = \{v_i\}_{i=1}^n \), where the last \( n - k \) basis vectors are a basis for \( \ker(A) \) and the first \( k \) basis vectors are arbitrarily chosen to complete the basis. Then we let the basis for \( V \) be \( B_V = \{v_i\}_{i=1}^m \) where the first \( k \) basis vectors are defined by \( v_i = A(v_i) \) and the remaining \( m - k \) are arbitrarily chosen to complete the basis. Thus the block sizes are as follows:

\[
A = \begin{bmatrix}
I_{k \times k} & 0_{k \times (n-k)} \\
0_{(m-k) \times k} & 0_{(m-k) \times (n-k)}
\end{bmatrix}.
\]

Problem 5. The \( j \)-th column of the matrix representation of a linear operator \( A \) between finite-dimensional vector spaces is the operator applied to the \( j \)-th basis vector of the domain, expressed with respect to the basis of the codomain. Here, we have that the domain and the codomain are the same, and they are represented by the given basis \( \{v_i\}_{i=1}^n \). The problem description tells us that \( Av_1 = \lambda v_1 \), and that \( Av_k = \lambda v_k + v_{k-1} \) for \( k = 2, \ldots, n \). With this information, we see that the first column will contain the coordinates of \( \lambda v_1 \) expressed with respect to \( \{v_i\}_{i=1}^n \), which is simply \([\lambda \ 0 \ 0 \ \cdots \ 0]^T\). The other columns will continue to have the value \( \lambda \) at the diagonal elements, preceded by the number 1 on the super diagonal.

\[
A = \begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \ddots & \vdots \\
0 & 0 & \lambda & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & \cdots & 0 & \lambda
\end{bmatrix}
\]

This type of matrix representation is known as the “Jordan Form,” which we will discuss in detail later on in this class.

Problem 6. First observe that \( \text{rk}(AB) = AB(\mathbb{R}^p) = A(\text{rk}(B)) = \text{rk}(A|_{\text{rk}(B)}) \subseteq \text{rk}(A) \), so \( \text{rk}(AB) \leq \text{rk}(A) \). Similarly, \( \text{rk}(\ker(B)) = A(\ker(B)) = A(0) = 0 \), so \( \ker(AB) \supseteq \ker(B) \) and consequently \( \text{nl}(AB) \geq \text{nl}(B) \).

By the rank-nullity theorem,

\[
p = \text{rk}(B) + \text{nl}(B) = \text{rk}(AB) + \text{nl}(AB)
\]

and therefore \( \text{rk}(AB) = \text{rk}(B) - (\text{nl}(AB) - \text{nl}(B)) \leq \text{rk}(B) \). Hence we have shown that \( \text{rk}(AB) \leq \min(\text{rk}(A), \text{rk}(B)) \).

To show the other side of the inequality, applying the rank-nullity theorem to \( A|_{\text{rk}(B)} \) yields

\[
\text{rk}(B) = \dim(\text{rk}(B)) = \text{rk}(A|_{\text{rk}(B)}) + \text{nl}(A|_{\text{rk}(B)}) = \text{rk}(AB) + \text{nl}(A|_{\text{rk}(B)})
\]
Now, again using the rank-nullity theorem, we have \( \text{nl}(A|_{R(B)}) \leq \text{nl}(A) = n - \text{rk}(A) \). Therefore

\[
\text{rk}(B) \leq \text{rk}(AB) + n - \text{rk}(A)
\]

and we have shown the second inequality.

**Problem 7.** Let \( A \in \mathbb{R}^{m \times n} \). Also, let \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)} \) be the singular values of \( A \). Using the SVD theorem, \( A \) can be decomposed as \( U \Sigma V^T \), where \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{n \times n} \) are unitary matrices and \( \Sigma \in \mathbb{R}^{m \times n} \) is a diagonal matrix containing the singular values of \( A \) along the principal diagonal. Consider any vector \( x \in \mathbb{R}^n \).

\[
\|Ax\|_2 = \sqrt{(Ax)^T(Ax)}
\]

\[
= \sqrt{x^T(A^T A)x}
\]

\[
= \sqrt{x^T V \Sigma_1^2 V^T x},
\]

where \( \Sigma_1 \in \mathbb{R}^{n \times n} \) is a diagonal matrix whose diagonal entries are given by \( \{\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)}\} \) if \( n \leq m \) and by \( \{\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m, 0, \ldots, 0\} \) otherwise.

Now, since \( V \) is a unitary matrix, its columns form a basis for \( \mathbb{R}^n \); therefore, vector \( x \) can be written as a linear combination of these columns. Consequently, \( x = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \), where \( \alpha_i \in \mathbb{R} \) and \( v_i \) denotes the \( i \)th column of \( V \). Therefore, \( V^T x = \alpha_1 V^T v_1 + \ldots + \alpha_n V^T v_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)^2 \). From (3), we have:

\[
\|Ax\|_2 = \sqrt{\sigma_1^2 \alpha_1^2 + \ldots + \sigma_{\min(m,n)}^2 \alpha_{\min(m,n)}^2}
\]

\[
\leq \sigma_1 \sqrt{\alpha_1^2 + \ldots + \alpha_n^2}
\]

\[
= \sigma_1 \|x\|_2 \ \forall x \in \mathbb{R}^n.
\]

Therefore,

\[
\|A\|_{2,i} := \sup_{\|x\|=1, x \in \mathbb{R}^n} \|Ax\|_2 \leq \sigma_1 \|x\|_2.
\]

However, choosing \( x = v_1 \) implies that (6) holds with equality. Thus, \( \|A\|_{2,i} = \sigma_1 \).