EE221A Section 3

September 15, 2017

1 Recap

Draw diagrams reviewing maps, null space, range space, bases, surjective, injective, etc.
2 Norms

Let \((V, F)\) be a vector space with \(F = \mathbb{R}\) or \(F = \mathbb{C}\). A norm on that space is a map \(\|\cdot\| : V \to \mathbb{R}_+\) satisfying the following axioms:

1. \(\|\alpha v\| = |\alpha| \|v\|\) \(\forall \alpha \in F, v \in V\) (absolute homogeneity)
2. \(\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|\) \(\forall v_1, v_2 \in V\) (subadditivity or triangle inequality)
3. \(\|v\| = 0 \iff v = \theta_V\)

Consider. We define the codomain of \(\|\cdot\|\) as \(\mathbb{R}_+\). Do the axioms above imply that \(\|\cdot\|\) always produces a non-negative result?

Consider. Why are norms useful?

Definition 1 (Equivalent Norms). Two norms \(\|\cdot\|_a\) and \(\|\cdot\|_b\) on \((V, F)\) are said to be equivalent if \(\exists m_l, m_u \in \mathbb{R}_+\) such that \(\forall v \in V:\)

\[m_l \|v\|_a \leq \|v\|_b \leq m_u \|v\|_a\]

Fact 1. All norms over finite dimensional vectors spaces are equivalent.

Problem 1. Find values for \(m_l\) and \(m_u\) in the case where \(\|\cdot\|_a = \|\cdot\|_\infty\) and \(\|\cdot\|_b = \|\cdot\|_1\)

Consider. Why is it helpful to have equivalent norms on a vector space?
3 Induced norms

Let $U$ and $V$ be normed linear spaces with norms $\| \cdot \|_U$ and $\| \cdot \|_V$, respectively.

**Definition 2.** The induced norm of a continuous linear operator $A : U \to V$ is given by

$$\|A\|_i := \sup_{u \neq \theta_U} \frac{\|A(u)\|_V}{\|u\|_U} = \sup_{\|u\|_U = 1} \|A(u)\|_V$$

**Consider.** Norms on two vector spaces $U$ and $V$ “induce” a norm on the space of linear maps between $U$ and $V$. (**Exercise:** Prove that the space of linear maps between two vector spaces is a vector space.) In fact, every pair of norms $(\| \cdot \|_U, \| \cdot \|_V)$ may induce a different norm on that space.

**Consider.** Why do we need continuity of a linear map to define an induced norm?

**Fact 2.** All linear maps over finite dimensional vectors spaces are continuous.

**Proposition 3.** Let $U, V, W$ be normed linear spaces. Given the linear operators $A, \tilde{A} : V \to W$, and $B : U \to V$, we have the following properties:

1. $\|A(v)\|_W \leq \|A\|_i \|v\|_V$
2. $\|\alpha A\|_i = |\alpha| \|A\|_i$
3. $\|A + \tilde{A}\|_i \leq \|A\|_i + \|\tilde{A}\|_i$
4. $\|A\|_i = 0 \iff A = 0$
5. $\|AB\|_i \leq \|A\|_i \|B\|_i$

**Problem 2.** Prove the above properties.

**Problem 3** (Induced norms of matrices). Given a linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$, we let $A \in \mathbb{R}^{m \times n}$ be its matrix representation. Then the induced norm of $A$ is defined by

$$\|A\|_{p,i} := \sup_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p}$$

Show that

$$\|A\|_{\infty,i} = \max_{1 \leq i \leq m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} \quad \text{(max absolute row sum)}$$

**Consider.** Why are induced norms useful?
4 Inner product spaces and Hilbert spaces

Definition 4. Consider a vector space $(H, \mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. The function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$ is called an inner product if

1. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
2. $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$, $\forall \alpha \in \mathbb{F}$
3. $\|x\|^2 = \langle x, x \rangle \geq 0$, and $\|x\|^2 = 0 \iff x = 0$
   (here $\|x\|$ is referred to as the norm induced by the inner product)
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Definition 5. A space equipped with an inner product is called an inner product space.

Definition 6. A space equipped with a norm is called a normed space.

Definition 7. A space is complete if every Cauchy sequence of elements in that space converges to an element in that space. A Cauchy sequence is any sequence $x_1, x_2, \ldots$ such that

$\forall \varepsilon > 0, \exists N$ such that $\|x_m - x_n\| < \varepsilon, \forall m, n > N$

Definition 8. A complete inner product space is called a Hilbert space (here completeness is w.r.t the norm induced by the inner product).

Definition 9. A complete normed space is called a Banach space (here completeness is w.r.t any norm).

Proposition 10.

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle$
3. $\langle \alpha x + \beta y, \gamma z + \delta w \rangle = \overline{\alpha} \gamma \langle x, z \rangle + \overline{\beta} \gamma \langle y, z \rangle + \overline{\alpha} \delta \langle x, w \rangle + \overline{\beta} \delta \langle y, w \rangle$

Problem 4. Prove the above properties.
5 Orthogonality

Let $H$ be a Hilbert space.

Definition 11. Given $x, y \in H$, $x$ and $y$ are said to be orthogonal if $\langle x, y \rangle = 0$.

Definition 12. $M^\perp := \{ y \in H | \langle x, y \rangle = 0, \forall x \in M \}$ is called the orthogonal complement of $M$.

Exercise. Prove that $M^\perp$ is a vector space.

Problem 5 (Cauchy-Schwarz inequality). Let $H$ be a Hilbert space. Show that

$$|\langle a, b \rangle| \leq \|a\|\|b\|$$

6 Adjoint

Definition 13. Let $U, V$ be Hilbert spaces, and $A : U \to V$ be a continuous linear map. The map $A^* : V \to U$ is said to be the adjoint of $A$ if

$$\langle v, A(u) \rangle_V = \langle A^*(v), u \rangle_U$$

Exercise. Prove that $A^*$ is a linear map.

Problem 6. Let $A : U \to V$ be a linear map, where $U, V$ are Hilbert spaces. Show that

$$\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$$