EE221A Section 2
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1 Linear Maps

Let $U, V$ be vector spaces and let $F$ be a field.

**Definition 1.** A map $L : U \to V$ is said to be *linear* if $L(\alpha u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2)$ for all $u_1, u_2 \in U$ and for all $\alpha_1, \alpha_2 \in F$.

**Problem 1 (Linearity).** Consider a one-dimensional dynamical system of the form

$$
\dot{x}(t) = x(t) \quad \text{for } t \in [0, 1] \n$$

$$
x(0) = x_0,
$$

for some $x_0 \in \mathbb{R}$. Let $M(x_0) : \mathbb{R} \to \mathbb{R}$ be the map that maps the initial state $x_0$ to the final state $x(1)$, i.e., $M(x_0) = x(1)$. Is $M$ linear?

Now consider the same setup except that $\dot{x}(t) = x(t) + u$, $u \in \mathbb{R}$. Is the map $Q(x_0, u) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ linear?

2 Matrix Representation

**Fact:** Any linear map over finite dimensional vector spaces can be represented by matrix multiplication.

**Basic Idea:** Let $U, V$ be finite dimensional vector spaces and let $A : U \to V$ be a linear map. Consider bases $\{u_j\}_{j=1}^n$ of $U$ and $\{v_i\}_{i=1}^m$ of $V$. For each $j = 1, \ldots, n$, there exists a unique $\{a_{ij}\}_{i=1}^m$ such that

$$
A(u_j) = \sum_{i=1}^m a_{ij} v_i.
$$

Then we can define an $m \times n$ matrix $A$, whose $(i, j)$ th element is $a_{ij}$. In other words, the $j$th column of $A$ is $A(u_j)$ expressed with respect to $\{v_i\}_{i=1}^m$.

**Matrix multiplication on coordinate vectors:** For the coordinate (basis weight) $\eta \in \mathbb{F}^m$ of $A(x)$, and the coordinate $\xi \in \mathbb{F}^n$ of $x$, we have $\eta = A\xi$. 
Problem 2 (Matrix representation). Let $\mathbb{R}^n[s]$ be the vector space of polynomials in $s$ whose degree is less than or equal to $n$. Consider a linear map $A : \mathbb{R}^n[s] \to \mathbb{R}^{n-1}[s]$ with $A(u) = \frac{du}{ds}$. Define bases of the domain and codomain, and give the matrix representation of $A$.

3 Null Space

Definition 2. Given a linear map $A : U \to V$, we call $\mathcal{N}(A) := \{ u \in U \mid A(u) = 0_V \}$ the null space of $A$. Define nullity$(A) := \dim \mathcal{N}(A)$

The nullspace of a linear map $\mathcal{N}(A)$ is:

- the set of vectors mapped to zero by $y = A(x)$
- the set of vectors orthogonal to all rows of $A$
- the source of ambiguity in $x$ given $y = A(x)$
  - if $y = A(x)$ and $z \in \mathcal{N}(A)$, then $y = A(x + z)$
  - if $y = A(x)$ and $y = A(\tilde{x})$, then $\tilde{x} = x + z$ for some $z \in \mathcal{N}(A)$

Fact: A linear map $A$ is called injective (one-to-one) if $0_V$ is the only element of its nullspace. $\mathcal{N}(A) = 0_V$ iff:

- $x$ can always be uniquely determined from $y = A(x)$ (i.e. linear transformation $y = A(x)$ doesn’t "lose" information)
- mapping from $x$ to $A(x)$ is injective: different $x$'s map to different $y$'s
- columns of the matrix $A$ are independent (and therefore form a basis for $A$)
- $A$ has a left inverse
- $\det(A^T A) \neq 0$
## 4 Range Spaces

**Definition 3.** Given a linear map \( A : U \to V \), we call \( R(A) := \{ v \in V \mid v = A(u), \ u \in U \} \) the range space of \( A \).

Define \( \text{rank}(A) := \dim(R(A)) \)

The range of a linear map \( R(A) \) is:

- the set of vectors that can be actually reached by linear mapping \( y = A(x) \)
- the span of the columns of matrix \( A \)
- the set of vectors \( y \) for which \( y = A(x) \) has a solution

**Fact:** A linear map \( A \) is called surjective (onto) if \( R(A) = V \). This happens iff:

- \( Ax = y \) can be solved in \( x \) for any \( y \)
- rows of \( A \) are independent
- \( A \) has a right inverse
- \( \mathcal{N}(A) = \theta_V \)
- columns of matrix \( A \) span \( V \)
- \( \det(AA^T) \neq 0 \)

**Recall.** What determines the dimensionality of a space?

**Problem 3** (Rank-Nullity Theorem). For a linear map \( L : U \to V \), let \( n \) be the dimension of \( U \). Prove that \( \dim(R(L)) + \dim(\mathcal{N}(L)) = n \).

## 5 Invertibility

**Fact:** A linear map \( A : \mathbb{R}^n \to \mathbb{R}^n \) is invertible or nonsingular if \( \det(A) \neq 0 \). Equivalent conditions:

- columns of \( A \) are a basis for \( \mathbb{R}^n \)
- rows of \( A \) are a basis for \( R^n \)
- \( y = Ax \) has a unique solution \( x \) for every \( y \in \mathbb{R}^n \)
- \( A \) has a (left and right) inverse denoted \( A^{-1} \in \mathbb{R}^{n \times n} \), \( AA^{-1} = A^{-1}A = I \)
- \( \mathcal{N}(A) = \theta \)
- \( R(A) = \mathbb{R}^n \)
- \( \det(A^TA) = \det(AA^T) \neq 0 \)
- \( x = A^{-1}y \) is the unique solution of \( Ax = y \)
Problem 4 (Injectivity and Surjectivity). Recall that a function can be injective, surjective, both (bijective), or neither. Explain the relationship between rank, nullity, injectivity, and surjectivity.

Problem 5 (Deriving range and nullspace). Let $\mathbb{R}^n[s]$ be the vector space of polynomials in $s$ whose degree is less than or equal to $n$. Consider a linear map $L : \mathbb{R}^n[s] \to \mathbb{R}^n[s]$ with $L(u) = \frac{du}{ds}$. Specify the range and null space of $L$ and their dimensions. Is this linear map bijective?