1 Subspaces

**Definition 1.** $W$ is said to be a subspace of a vector space $V$ over a field $F$ if $W \subseteq V$ and $W$ is itself a vector space.

For proving that $W$ is a subspace of $V$, if we know that $W$ is a subset of $V$, then all that remains is to verify that

1. $w_1, w_2 \in W \implies w_1 + w_2 \in W$
2. $\forall \alpha \in F, \; \alpha \cdot w \in W$

Consider $(C(\mathbb{R}, \mathbb{R}), \mathbb{R})$ the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (which we have previously established as a vector space). Let’s define the set of linear functions as

$$L = \{l(\cdot) : l(x) = \alpha x \text{ for some } \alpha \} \text{ (not all } l(\cdot) \text{ need the same } \alpha)$$

We can show that $L$ is a subspace of $C$ by showing that linear combinations of functions in $L$ result in a function in $L$. Similarly, we can show that the set of affine functions

$$A = \{a(\cdot) : a(x) = \alpha x + \beta \text{ for some } \alpha, \beta \} \text{ (not all } a(\cdot) \text{ need the same } \alpha, \beta)$$

is also a subspace of $C$.

2 Functions and Matrix Rank

**Definition 2 (Rank Nullity Theorem).** For a linear map $A : U \to V$, we have:

$$\text{rank}(A) + \text{dim}(N(A)) = \text{dim}(U)$$

**Remark 1.** Rank-Nullity theorem is a powerful result in linear algebra and allows us to bound the rank of linear maps (and their compositions as we will see below in Sylvester’s Inequality.)

**Remark 2.** The proof of Rank-Nullity theorem is also very insightful. In particular, suppose that $\{y_1, \ldots, y_k\}$ is a basis of $N(A)$, then we can extend this basis as $\{y_1, \ldots, y_k, z_1, \ldots, z_{n-k}\}$ to get a basis for $U$, where $n = \text{dim}(U)$. The proof hints that the range of $A$ can be spanned by $\{A(z_1), \ldots, A(z_{n-k})\}$. Moreover, these vectors are linearly independent in $V$ and hence form a basis for $R(A)$.

**Definition 3 (Sylvester’s Inequality).** For two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, we can bound the rank of their product $AB$ by the following inequality

$$\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$
We recall that in order for a linear map \( A : U \to V \) to be surjective, its range must equal its codomain \( (R(A) = V) \). For an operator to be injective, it must have a trivial nullspace \( (N(A) = \{0_U\}) \). In terms of matrix rank, this means that \( \text{rank}(A) = \dim(V) \) (for surjectivity) and \( \text{rank}(A) = \dim(U) \) (for injectivity).

We can represent compositions of matrix operators as multiplication of their matrix representations, so we can use Sylvester’s inequality to reason about the possible injectivity or surjectivity of a composition of linear operations.

Remark 3. Note that Sylvester’s inequality can be very conveniently proved by noting that

\[
AB : \mathbb{R}^p \to \mathbb{R}^m \equiv A : R(B) \to \mathbb{R}^m,
\]

and then using the rank-nullity theorem on the map \( A \) defined above.

### 3 Matrix Representation of a Linear Map

For a linear map \( A : U \to V \), we can obtain the matrix representation \( A \) of \( A \) using the following 5-step algorithm:

1. Identify the basis \( B_U = \{u_1, \ldots, u_n\} \) for the domain \( U \).
2. Identify the basis \( B_V = \{v_1, \ldots, v_m\} \) for the co-domain \( V \).
3. For each element \( u_i \in B_U \), determine \( y_i := A(u_i) \).
4. Represent each \( y_i \) in the basis \( B_V \) of co-domain. Let the co-ordinates obtained for \( y_i \) through this process are given by the vector \( \alpha_i \in \mathbb{R}^m \).
5. The \( i \)th column of \( A \) is then given by \( \alpha_i \).

### 4 Change of Basis

Consider \( \{u_i\}_{i=1}^n, \{\tilde{u}_i\}_{i=1}^n \) : two bases of \( U \).
\( \{v_j\}_{j=1}^m, \{\tilde{v}_j\}_{j=1}^m \) : two bases of \( V \).

\( A \): matrix representation of \( A \) with \( \{u_i\}_{i=1}^n, \{v_j\}_{j=1}^m \).
\( \tilde{A} \): matrix representation of \( A \) with \( \{\tilde{u}_i\}_{i=1}^n, \{\tilde{v}_j\}_{j=1}^m \).

\( R \) (or \( \tilde{R} \)): \( n \times n \) matrix whose \( i \)th column is \( u_i \) (or \( \tilde{u}_i \)).
\( S \) (or \( \tilde{S} \)): \( m \times m \) matrix whose \( j \)th column is \( v_j \) (or \( \tilde{v}_j \)).

For any \( x \in U, x = RX = \tilde{R}\tilde{x} \implies \xi = P\tilde{\xi} \), where \( P = R^{-1}\tilde{R} \).
For \( y = A(x) \in V, y = SY = \tilde{S}\tilde{y} \implies \tilde{\eta} = Q\eta \), where \( Q = \tilde{S}^{-1}S \).
In addition, \( \eta = A\xi \).

\[ \therefore \tilde{\eta} = QAP\tilde{\xi} \implies \tilde{A} = QAP \]
Remark 4. One can also directly determine $\bar{A}$ using the 5-step algorithm described in Section 3.

5 Singular Value Decomposition

To solve an SVD problem:
1. Solve for $V$ (or $U$) using $AA^\top$ (or $A^\top A$)
2. Solve for the other unitary matrix using $\sigma_i u_i = Av_i$

6 Lipschitz Continuity

- Recall flow chart from discussion for when we can prove/disprove global and local lipschitz properties
- Just because we haven’t proven something is lipschitz doesn’t mean there is no unique solution.

7 Bellman-Gronwall Lemma

**Theorem 4.** Let $u(\cdot)$ be a nonnegative, piecewise continuous function on $[0, T]$ which satisfies

$$u(t) \leq C_1 + \int_{t_0}^t k(\tau)u(\tau)d\tau$$

for some constant $C_1 \geq 0$ and a nonnegative integrable function $k$. Then

$$u(t) \leq C_1 \exp \left( \int_{t_0}^t k(\tau)d\tau \right),$$

for $0 \leq t_0 < t \leq T$.

**Remark 5.** The above result holds as it is if $C_1$ is a non-negative function of time.

**Remark 6.** Bellman-Gronwall lemma is very useful for the perturbation analysis of a system. We did examples of this in discussion and in the homework.
8 Dynamical Systems

\((U, Y, \Sigma, s, r): \) (input, state, output, state transition function, output read-out map).

- **Input:** \(U \subset \{ u : [0, \infty) \rightarrow U \} \) (Note that \(U\) is a function space.)
- **Output:** \(Y \subset \{ y : [0, \infty) \rightarrow Y \} \) (Note that \(Y\) is a function space.)
- **State Space:** \(\Sigma\), a vector space (typically \(\mathbb{R}^n\))
- **State transition function:** \(s : \mathbb{R} \times \mathbb{R} \times \Sigma \times U \rightarrow \Sigma\) with \(s(t, t_0, x_0, u[t_0, t]) = x(t)\)
- **Output read-out map:** \(r : \mathbb{R} \times \Sigma \times U \rightarrow Y\) with \(r(t, x(t), u(t)) = y(t)\)
- **Response function:** composition of \(s\) and \(r\): \(\rho : \mathbb{R} \times \mathbb{R} \times \Sigma \times U \rightarrow Y\) with \(\rho(t, t_0, x_0, u[t_0, t]) = y(t)\)

9 The State-Transition Matrix \(\Phi\)

**Definition 5.** The matrix-valued function \(\Phi(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{n \times n}\) is called the state-transition matrix of \(A\) if \(\Phi(\cdot, t_0)\) solves the matrix differential equation

\[
\dot{X}(t) = A(t)X(t), \quad X(t) \in \mathbb{R}^{n \times n}
\]

\(X(t_0) = I\).

**Properties**

1. The solution of \(\dot{x} = A(t)x, s(t, t_0, x_0)\) is given by \(s(t, t_0, x_0) = \Phi(t, t_0)x_0\)
2. \(\forall t, t_0, t_1 \in \mathbb{R}^+, \Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)\)
3. \([\Phi(t, t_0)]^{-1} = \Phi(t_0, t)\)
4. \(\det \Phi(t, t_0) = \exp \int_{t_0}^{t} \text{trace}(A(\tau))d\tau\)

10 The Matrix Exponential

**Proposition 6 (Matrix Exponential).** The state-transition matrix for the system

\[
\dot{X}(t) = AX(t), \quad X(t) \in \mathbb{R}^{n \times n}
\]

\(X(t_0) = I\)

is the matrix exponential \(\Phi(t, t_0) = e^{A(t-t_0)} = \sum_{k=0}^{\infty} \frac{A^k(t-t_0)^k}{k!}\).