EECS192 Mechatronic Design Laboratory

Vehicle Steering Notes. Spring 2016

1 Bicycle Kinematics

The kinematic equations are given by:

$$\dot{x}_b = V cos(\theta(t)) \tag{1}$$

$$\dot{y}_b = -V sin(\theta(t)) \tag{2}$$

$$\dot{\theta} = \frac{V}{L} tan(\delta(t)) \tag{3}$$

$$y_a = y_b - L sin(\theta(t)) \tag{4}$$

For simplicity, we can assume that the vehicle speed V is constant. There is then just one control input the system, the steering angle δ, and we can consider the output to be y_a, the road distance from the front axle. Now for following a straight track with a small heading error (say less than 20°), we can linearize the differential equations using $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Thus we get:

$$\dot{y}_b \approx -V\theta \tag{5}$$

$$\dot{\theta} \approx \frac{V}{L}\delta(t) \tag{6}$$

$$\dot{y}_a \approx \dot{y}_b - L\dot{\theta} = -V\theta - L\dot{\theta} \tag{7}$$

We would like to get a differential equation relating the input steering angle to the front axle position error. To do this, we differentiate eqn. 7 and substitute eqn. 6 for steering angle obtaining

$$\ddot{y}_a = \frac{-V^2}{L}\delta(t) - V\dot{\delta}(t). \tag{8}$$

Table 1: Definition of Variables

Variable	Description
x_b	X coordinate of midpoint of rear axle
x_a	X coordinate of midpoint of front axle
y_b	lateral displacement w.r.t. road centerline at rear axle
y_a	lateral displacement w.r.t. road centerline at front axle
δ	steering angle
L	wheel base
θ	relative yaw angle w.r.t. road centerline
V	vehicle speed

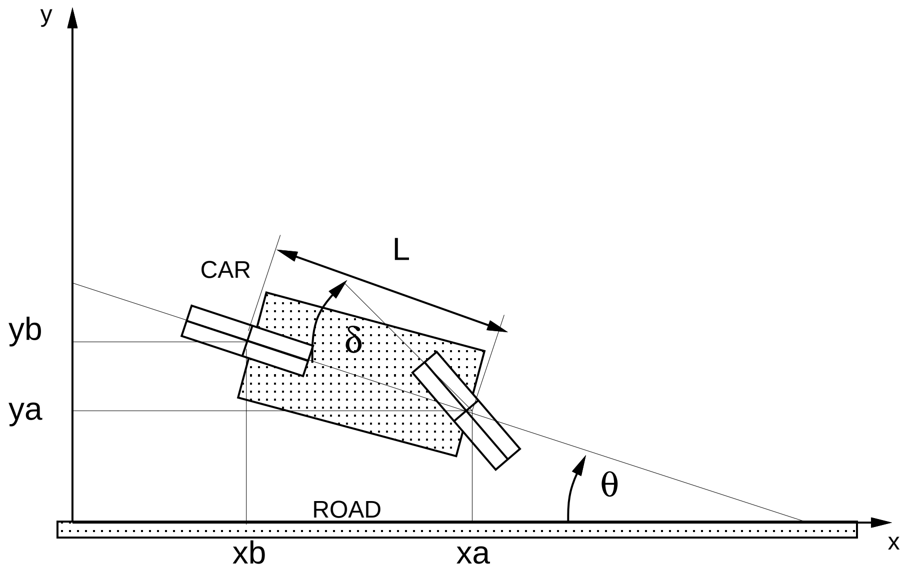

Figure 1: Bicycle Model for Steering Kinematics

2 Proportional Control

Let's see what happens when we apply a steering control to the system proportional to position error:

$$\delta(t) = k_p y_a(t) \tag{9}$$

Then the closed loop system has dynamics described by the second order linear differential equation:

$$\ddot{y}_a + V k_p \dot{y}_a(t) + \frac{V^2}{L} k_p y_a(t) = 0. \tag{10}$$

Let's re-write this second order differential equation in state variable form, letting $x_1 = y_a$ and $x_2 = \dot{y}_a$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-V^2}{L} k_p & -V k_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

This is just a homogeneous equation of the form $\dot{\mathbf{x}} = A\mathbf{x}$, so we know the solution is just:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) \tag{12}$$

where $e^A t$ is a matrix exponential given by

$$e^A t = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ... \tag{13}$$

The system will be stable if the real part of the eigenvalues of A are less than 0. You can verify that the eigenvalues of A are

$$\lambda_{1,2} = \frac{V}{2}\left(-k_p \pm \sqrt{k_p^2 - \frac{4k_p}{L}}\right) \tag{14}$$