In this example we review discretization, controllability, and minimum norm solutions. Consider the model of a car moving in a lane

\[
\frac{dp(t)}{dt} = v(t) \\
\frac{dv(t)}{dt} = \frac{1}{RM} u(t)
\]

where \(p(t)\) is position, \(v(t)\) is velocity, \(u(t)\) is wheel torque, \(R\) is wheel radius, and \(M\) is mass. This model is similar to an example discussed in Lecture 7A, but here we ignore friction for simplicity.

First we discretize this continuous-time model. If we apply the constant input \(u(t) = u_d(k)\) from \(t = kT\) to \((k + 1)T\), then by integration

\[
v(t) = v(kT) + (t - kT) \frac{1}{RM} u_d(k) \\
p(t) = p(kT) + (t - kT)v(kT) + \frac{1}{2} (t - kT)^2 \frac{1}{RM} u_d(k)
\]

for \(t \in [kT, (k + 1)T]\). In particular, at \(t = (k + 1)T\:

\[
p((k + 1)T) = p(kT) + Tv(kT) + \frac{T^2}{2RM} u_d(k) \\
v((k + 1)T) = v(kT) + \frac{T}{RM} u_d(k).
\]

Putting these equations in matrix/vector form and substituting \(p_d(k) = p(kT), v_d(k) = v(kT)\), we get

\[
\begin{bmatrix}
  p_d(k+1) \\
  v_d(k+1)
\end{bmatrix} = 
\begin{bmatrix}
  1 & T \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  p_d(k) \\
  v_d(k)
\end{bmatrix} + 
\begin{bmatrix}
  \frac{1}{2} T^2 \\
  \frac{T}{RM}
\end{bmatrix}
\begin{bmatrix}
  u_d(k)
\end{bmatrix}.
\]  

(1)

Now suppose the vehicle is at rest with \(p(0) = v(0) = 0\) at \(t = 0\) and the goal is to reach a target position \(p_{\text{target}}\) and stop there \((v_{\text{target}} = 0)\). Recall from the lectures on controllability that if we can find a sequence \(u_d(0), u_d(1), \ldots, u_d(\ell - 1)\) such that

\[
\begin{bmatrix}
  p_{\text{target}} \\
  0
\end{bmatrix} = 
\begin{bmatrix}
  \vec{b} \\
  \vec{0}
\end{bmatrix}
\begin{bmatrix}
  A\vec{b} \\
  A^2\vec{b} \\
  \vdots \\
  A^{\ell-1}\vec{b}
\end{bmatrix}
\begin{bmatrix}
  u_d(\ell - 1) \\
  u_d(\ell - 2) \\
  \vdots \\
  u_d(0)
\end{bmatrix}
\]

(2)
then we reach the desired state in $\ell$ time steps, that is at time $t = \ell T$.

Since we have $n = 2$ state variables the controllability test we learned checks whether $C_{\ell}$ with $\ell = 2$ spans $\mathbb{R}^2$. This is indeed the case, since

$$C_2 = \begin{bmatrix} \vec{b} & \vec{A} \vec{b} \end{bmatrix} = \frac{1}{RM} \begin{bmatrix} \frac{1}{2}T^2 & \frac{3}{2}T^2 \end{bmatrix}$$

has linearly independent columns.

Although this test also suggests we can reach the target state in two steps, the resulting values of $u_{d}(0)$ and $u_{d}(1)$ will likely exceed physical limits. For example, if we take the values\(^2\) $RM = 5000$ kg m, $T = 0.1$ s, $p_{\text{target}} = 1000$ m, then

$$\begin{bmatrix} u_{d}(1) \\ u_{d}(0) \end{bmatrix} = C_{2}^{-1} \begin{bmatrix} p_{\text{target}} \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \cdot 10^8 \\ 5 \cdot 10^8 \end{bmatrix} \text{kg m}^2/\text{s}^2,$$

which exceeds the torque and braking limits of a typical car by 5 orders of magnitude.\(^3\)

Therefore, in practice we need to select a sufficiently large number of time steps $\ell$. This leads to a wide controllability matrix $C_{\ell}$ and allows for infinitely many input sequences that satisfy (2). Among them we can select the minimum norm solution so we spend the least control energy. Using the minimum-norm solution formula

$$\begin{bmatrix} u_{d}(\ell - 1) \\ u_{d}(\ell - 2) \\ \vdots \\ u_{d}(0) \end{bmatrix} = C_{\ell}^T (C_{\ell} C_{\ell}^T)^{-1} \begin{bmatrix} p_{\text{target}} \\ 0 \end{bmatrix}$$

and quite a bit of algebra, one will obtain the input sequence

$$u_{d}(k) = \frac{6RM(\ell - 1 - 2k)}{T^2(\ell^2 - 1)} p_{\text{target}}, \quad k = 0, \ldots, \ell - 1.$$ 

In the plot below we show this input sequence, as well as the resulting velocity and position profiles for $RM = 5000$ kg m, $p_{\text{target}} = 1000$ m, $T = 0.1$ s, and $\ell = 1200$. With these parameters we allow $\ell T = 120$ s (2 minutes) to travel 1 km. Note that the vehicle accelerates in the first half of this period and decelerates in the second half, reaching the maximum velocity 12.5 m/s ($\approx 28$ mph) in the middle. The acceleration and deceleration are hardest at the very beginning and at the very end, respectively. The corresponding torque is within a physically reasonable range, $[-2000, 2000]$ Nm.

\(^2\) say, for a sedan with mass $M \approx 1700$ kg and wheel radius $R \approx 0.3m$

\(^3\) If our car could deliver the torque $u_{d}(0) = 5 \cdot 10^8$ kg m$^2$/s$^2$, then from (1) we would reach $v_{d}(1) = v(T) = 10^4$ m/s (22,369 mph) in $T = 0.1$ seconds!
Stability of Linear State Models

The Scalar Case

We first study a system with a single state variable $x(t)$ that obeys

$$x(t + 1) = \lambda x(t) + bu(t)$$

(3)

where $\lambda$ and $b$ are constants. If we start with the initial condition $x(0)$, then we get by recursion

$$x(1) = \lambda x(0) + bu(0)$$
$$x(2) = \lambda x(1) + bu(1) = \lambda^2 x(0) + \lambda bu(0) + bu(1)$$
$$x(3) = \lambda x(2) + bu(2) = \lambda^3 x(0) + \lambda^2 bu(0) + \lambda bu(1) + bu(2)$$

$$\vdots$$

$$x(t) = \lambda^t x(0) + \lambda^{t-1} bu(0) + \lambda^{t-2} bu(1) + \cdots + \lambda bu(t-2) + bu(t-1),$$

rewritten compactly as:

$$x(t) = \lambda^t x(0) + \sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k) \quad t = 1, 2, 3, \ldots$$

(4)

The first term $\lambda^t x(0)$ represents the effect of the initial condition and the second term $\sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k)$ represents the effect of the input sequence $u(0), u(1), \ldots, u(t-1).$
**Definition.** We say that a system is *stable* if its state $x(t)$ remains bounded for any initial condition and any bounded input sequence. Conversely, we say it is *unstable* if we can find an initial condition and a bounded input sequence such that $|x(t)| \to \infty$ as $t \to \infty$.

It follows from (4) that, if $|\lambda| > 1$, then a nonzero initial condition $x(0) \neq 0$ is enough to drive $|x(t)|$ unbounded. This is because $|\lambda|^t$ grows unbounded and, with $u(t) = 0$ for all $t$, we get $|x(t)| = |\lambda^t x(0)| = |\lambda|^t |x(0)| \to \infty$. Thus, (3) is unstable for $|\lambda| > 1$.

Next, we show that $|\lambda| < 1$ guarantees stability. In this case $\lambda^t x(0)$ decays to zero, so we need only to show that the second term in (4) remains bounded for any bounded input sequence. A bounded input means we can find a constant $M$ such that $|u(t)| \leq M$ for all $t$. Thus,

$$\left| \sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k) \right| \leq \sum_{k=0}^{t-1} |\lambda|^{t-1-k} |b||u(k)| \leq |b|M \sum_{k=0}^{t-1} |\lambda|^{t-1-k}.$$

Defining the new index $s = t - 1 - k$ we rewrite the last expression as

$$|b|M \sum_{s=0}^{t-1} |\lambda|^s,$$

and note that $\sum_{s=0}^{t-1} |\lambda|^s$ is a geometric series that converges to $\frac{1}{1-|\lambda|}$ since $|\lambda| < 1$. Therefore, each term in (4) is bounded and we conclude stability for $|\lambda| < 1$.

**Summary:** The scalar system (3) is stable when $|\lambda| < 1$, and unstable when $|\lambda| > 1$.

When $\lambda$ is a complex number, a perusal of the stability and instability arguments above show that the same conclusions hold if we interpret $|a|$ as the modulus of $a$, that is:

$$|\lambda| = \sqrt{\text{Re} \{\lambda\}^2 + \text{Im} \{\lambda\}^2}.$$

What happens when $|\lambda| = 1$? If we disallow inputs ($b = 0$), this case is referred to as “marginal stability” because $|\lambda^t x(0)| = |x(0)|$, which neither grows nor decays. If we allow inputs ($b \neq 0$), however, we can find a bounded input to drive the second term in (4) unbounded. For example, when $\lambda = 1$, the constant input $u(t) = 1$ yields:

$$\sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k) = \sum_{k=0}^{t-1} b = bt$$

which grows unbounded as $t \to \infty$. Therefore, $|\lambda| = 1$ is a precarious case that must be avoided in designing systems.
The Vector Case

When $\vec{x}(t)$ is an $n$-dimensional vector governed by
\[ \vec{x}(t + 1) = A\vec{x}(t) + Bu(t), \tag{5} \]
recursive calculations lead to the solution
\[ \vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k}Bu(k) \quad t = 1, 2, 3, \ldots \tag{6} \]
where the matrix power is defined as $A^t = A \cdots A$, $t$ times.

Since $A$ is no longer a scalar, stability properties are not apparent from (6). However, when $A$ is diagonalizable we can employ the change of variables $\vec{z} := T\vec{x}$ and select the matrix $T$ such that
\[ A_{\text{new}} = TAT^{-1} \]
is diagonal. $A$ and $A_{\text{new}}$ have the same eigenvalues and, since $A_{\text{new}}$ is diagonal, the eigenvalues appear as its diagonal entries:
\[ A_{\text{new}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \]
The state model for the new variables is
\[ \vec{z}(t + 1) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \vec{z}(t) + B_{\text{new}}u(t), \quad B_{\text{new}} = TB, \tag{7} \]
which nicely decouples into scalar equations:
\[ z_i(t + 1) = \lambda_i z_i(t) + b_iu(t), \quad i = 1, \ldots, n \tag{8} \]
where we denote by $b_i$ the $i$-th entry of $B_{\text{new}}$. Then, the results for the scalar case above imply stability when $|\lambda_i| < 1$ and instability when $|\lambda_i| > 1$.

For the whole system to be stable each subsystem must be stable, therefore we need $|\lambda_i| < 1$ for each $i = 1, \ldots, n$. If there exists at least one eigenvalue $\lambda_i$ with $|\lambda_i| > 1$ then we conclude instability because we can drive the corresponding state $z_i(t)$ unbounded.

Summary: The discrete-time system (5) is stable if $|\lambda_i| < 1$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of $A$, and unstable if $|\lambda_i| > 1$ for some eigenvalue $\lambda_i$. 