This homework is optional. Problem 1 is needed for the lab. The problems will give you chance to practice for midterm.

Solutions will be published soon, after you have had some time to try out the problems.

1 Brain-machine interface

The iPython notebook pca_brain_machine_interface.ipynb will guide you through the process of analyzing brain machine interface data using principle component analysis (PCA). This will help you to prepare for the project, where you will need to use PCA as part of a classifier that will allow you to use voice or music inputs to control your car.

Please complete the notebook by following the instructions given.

Solution

The notebook pca_brain_machine_interface_sol.ipynb contains solutions to this exercise.

2 SVD from the other side

In lecture, we thought about the SVD for a matrix $A$ with $m$ rows and $n$ columns by looking at the $n \times n$ symmetric matrix $A^T A$ and its eigenbasis. This question is about seeing what happens when we look at the $m \times m$ symmetric matrix $Q = AA^T$ and its orthonormal eigenbasis $U = [\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_n]$ instead. Suppose we have sorted the eigenvalues so that the real eigenvalues $\lambda$ are sorted in descending order where $\lambda_i \tilde{u}_i = Q \tilde{u}_i$.

a) Show that $\lambda_i \geq 0$.

(HINT: You want to involve $\tilde{u}_i^T \tilde{u}_i$ somehow.)

Solution

We know that $u_i$ is an eigenvector of $Q$, so we can start with:

\[
\lambda_i \tilde{u}_i = Q \tilde{u}_i
\]
\[
\lambda_i \tilde{u}_i^T \tilde{u}_i = \tilde{u}_i^T Q \tilde{u}_i
\]
\[
\lambda_i \|\tilde{u}_i\|^2 = \tilde{u}_i^T Q \tilde{u}_i
\]
\[
\lambda_i \|\tilde{u}_i\|^2 = \tilde{u}_i^T AA^T \tilde{u}_i
\]
\[
\lambda_i \|\tilde{u}_i\|^2 = (A^T \tilde{u}_i)^T (A^T \tilde{u}_i)
\]
\[
\lambda_i \|\tilde{u}_i\|^2 = \|A^T \tilde{u}_i\|^2
\]
Since we know that any squared quantity is positive, the right side of the equation is positive. This means that $\lambda_i$ must also be positive.

b) Suppose that we define $\tilde{v}_i = \frac{A^T \tilde{u}_i}{\sqrt{\lambda_i}}$ for all $i$ for which $\lambda_i > 0$. Suppose that there are $\ell$ such eigenvalues. Show that $V = [\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_\ell]$ has orthonormal columns.

**Solution**

To show orthonormality, we can compute the inner product $\tilde{v}_i^T \tilde{v}_j$ for all $i, j \in \{1, 2, \ldots, \ell\}$ and demonstrate that $\tilde{v}_i^T \tilde{v}_j = 1$ if $i = j$ and $\tilde{v}_i^T \tilde{v}_j = 0$ if $i \neq j$:

$$\tilde{v}_i^T \tilde{v}_j = \frac{(A^T \tilde{u}_i)^T (A^T \tilde{u}_j)}{\sqrt{\lambda_i} \sqrt{\lambda_j}}$$

$$\tilde{v}_i^T \tilde{v}_j = \frac{\tilde{u}_i^T A A^T \tilde{u}_j}{\sqrt{\lambda_i} \sqrt{\lambda_j}}$$

If $i = j$ we saw that in the previous part $\lambda_i \|\tilde{u}_i\|^2 = \tilde{u}_i^T A A^T \tilde{u}_i$

$$\tilde{v}_i^T \tilde{v}_i = \frac{\lambda_i \|\tilde{u}_i\|^2}{\sqrt{\lambda_i} \sqrt{\lambda_i}} = 1$$

Since we know $U$ is orthonormal, $\|\tilde{u}_i\| = 1$:

$$\tilde{v}_i^T \tilde{v}_i = \frac{\lambda_i}{\lambda_i} = 1$$

If $i \neq j$, we know that $\tilde{u}_j$ is an eigenvector of $AA^T$, so the original equation becomes:

$$\tilde{v}_i^T \tilde{v}_j = \frac{\lambda_j \tilde{u}_i^T \tilde{u}_j}{\sqrt{\lambda_i} \sqrt{\lambda_j}}$$

Since $U$ is orthonormal, $\tilde{u}_i^T \tilde{u}_j = 0$ and so $\tilde{v}_i^T \tilde{v}_j = 0$.

c) **Prove that** $A = \sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_i \tilde{v}_i^T$.

(HINT: Take an arbitrary left-input $\tilde{x}^T$ and consider $\tilde{x}^T A$. Decompose $\tilde{x}$ into the $U$ basis. See what happens when you multiply things out.)
Solution

\[ A = \sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_i \tilde{v}_i^T \]

\[ A = \sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_i (A^T \tilde{u}_i)^T \]

\[ A = \sum_{i=1}^{\ell} \tilde{u}_i \tilde{v}_i^T A \]

\[ A = U_{\ell} U_{\ell}^T A \]

Here, \( U_{\ell} \) is the matrix with \( \ell \) columns that correspond to the non-zero eigenvalues. If \( U_{\ell} \) were square, then \( U_{\ell} U_{\ell}^T \) would be the identity and we would be done. But it isn’t square, and so we can’t stop here as far as a proof goes.

This is why the hint was given as it was given. The entire collection of \( \tilde{u}_i \) vectors do form a basis. So, we can simply look at \( \tilde{u}_j^T A = (A\tilde{u}_j)^T = (\sqrt{\lambda_j} \tilde{v}_j)^T \). Notice that if \( j \) is not between 1 and \( \ell \), this is zero. Let us compare this to \( \tilde{u}_j^T (\sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_i \tilde{v}_i^T) = \sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_j^T \tilde{u}_i \tilde{v}_i^T = \sqrt{\lambda_j} \tilde{v}_j^T \) if \( j \) is between 1 and \( \ell \), and zero otherwise by orthonormality. Since these are the same for all the elements of a basis, the matrices must be identical.

d) Consider an arbitrary input \( \vec{x} = \vec{x}_\perp + \sum_{i=1}^{\ell} a_i \vec{v}_i \) where \( \vec{x}_\perp \) is orthogonal to each of the \( \vec{v}_i \). Show that \( A\vec{x}_\perp = \vec{0} \).

(HINT: Use the previous part.)

Solution

\[ A\vec{x}_\perp = (\sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_i \tilde{v}_i^T) \vec{x}_\perp = \sum_{i=1}^{\ell} \sqrt{\lambda_i} \tilde{u}_i (\vec{v}_i^T \vec{x}_\perp) \]

The inner product of \( \vec{x}_\perp \) with each \( \vec{v}_i \) is being computed in this expression. Since we know that \( \vec{x}_\perp \) is orthogonal to all \( \vec{v}_i \), we know \( \vec{v}_i^T \vec{x}_\perp = 0 \) and thus

\[ A\vec{x}_\perp = \vec{0} \].
3 The Moore-Penrose Pseudoinverse for “Fat” Matrices

Suppose that we have a set of linear equations described as \( A\vec{x} = \vec{y} \). If \( A \) is invertible, we know that the solution is \( \vec{x} = A^{-1}\vec{y} \). However, what if \( A \) is not a square matrix? In EE16A, you saw how this problem could be approached for tall matrices \( A \) where it really wasn’t possible to find a solution that exactly matches all the measurements. The linear least-squares solution gives us a reasonable answer that asks for the “best” match in terms of reducing the norm of the error vector.

This problem deals with the other case — when the matrix \( A \) is short and fat. In this case, there are generally going to be lots of possible solutions — so which should we choose and why? We will walk you through the **Moore-Penrose pseudoinverse** that generalizes the idea of the matrix inverse and is derived from the singular value decomposition.

a) Suppose that you have the following matrix.

\[
A = \begin{bmatrix}
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]

Calculate the full SVD decomposition of \( A \). That is to say, calculate \( U, \Sigma, V \), such that

\[ A = U\Sigma V^T, \]

where \( U \) and \( V \) are unitary matrices.

Leave all work in exact form, not decimal.

*Note:* Do NOT use a computer to calculate the SVD.

**Solution**

\[
AA^\top = \begin{bmatrix}
3 & -1 \\
-1 & 3
\end{bmatrix}
\]

which has characteristic polynomial \( \lambda^2 - 6\lambda + 8 = 0 \), giving the eigenvalues 4 and 2. Solving \( A\vec{\lambda} = \lambda_i\vec{\sigma} \) produces eigenvectors \( \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \) and \( \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \) associated with eigenvalues 4 and 2 respectively.

The singular values are the square roots of the eigenvalues of \( AA^\top \), so

\[
\Sigma = \begin{bmatrix}
2 & 0 & 0 \\
0 & \sqrt{2} & 0
\end{bmatrix}
\]

and

\[
U = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}.
\]
We can then solve for the \( \mathbf{v} \) vectors using \( \mathbf{A}^T \mathbf{u}_i = \sigma_i \mathbf{v}_i \), giving \( \mathbf{v}_1 = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \) and \( \mathbf{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \). The last \( \mathbf{v} \) must be orthonormal to the other two, so we can pick \( \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \).

The SVD is:

\[
\mathbf{A} = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \sqrt{2} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0
\end{bmatrix}
\]

b) Let us think about what the SVD does. Let us look at matrix \( \mathbf{A} \) acting on some vector \( \mathbf{x} \) to give the result \( \mathbf{y} \). We have

\[\mathbf{A}\mathbf{x} = \mathbf{U}\Sigma\mathbf{V}^T\mathbf{x} = \mathbf{y}.\]

Observe that \( \mathbf{U} \) and \( \mathbf{V}^T \) are unitary matrices, so they cannot change the norm of the input vector while \( \Sigma \) scales the input vector. We will try to “reverse” these operations one at a time and then put them together.

If \( \mathbf{U} \) performs some transformation on the vector \( (\Sigma\mathbf{V}^T)\mathbf{x} \), what operator can we derive that will undo this?

**Solution**

By orthonormality, we know that \( \mathbf{U}^T\mathbf{U} = \mathbf{I} \). Therefore, \( \mathbf{U}^T \) undoes the transformation.

c) Derive a matrix that will “unscale,” or undo the effect of \( \Sigma \) where it is possible to undo. Recall that \( \Sigma \) has the same dimensions as \( \mathbf{A} \). Ignore any division by zeros (that is to say, let it stay zero).

**Solution**

If you observe the equation:

\[\Sigma\mathbf{x} = \mathbf{y},\]

you can see that \( \sigma_i x_i = y_i \) for \( i = 0, \ldots, m - 1 \), which means that to obtain \( x_i \) from \( y_i \), we need to multiply \( y_i \) by \( \frac{1}{\sigma_i} \). For any \( i > m - 1 \), the information in \( x_i \) is lost by multiplying with 0. Therefore, the reasonable guess for \( x_i \) is 0 in this case. That’s why we padded 0s in the bottom of \( \Sigma \) given below:
If \( \Sigma = \begin{bmatrix} \sigma_0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_{m-1} & 0 & \cdots & 0 \end{bmatrix} \), then \( \tilde{\Sigma} = \begin{bmatrix} \frac{1}{\sigma_0} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \).

d) Derive an operator that would undo the transformation by \( V^T \).

**Solution**

By orthonormality, we know that \( V^T V = VV^T = I \). Therefore, \( V \) undoes the transformation.

e) Try to use the previous parts to derive an “inverse” (which we will use \( A^\dagger \) to denote). That is to say,

\[
\tilde{x} = A^\dagger \tilde{y}.
\]

The reason why the word inverse is in quotes (or why this is called a pseudo-inverse) is because we’re ignoring the “divisions” by zero.

**Solution**

We can use the matrices we derived above to “undo” the effect of \( A \) and get the required solution. Of course, nothing can possibly be done for the information that was destroyed by the nullspace of \( A \) — there is no way to recover any component of the true \( \tilde{x} \) that was in the nullspace of \( A \). However, we can get back everything else.

\[
\tilde{y} = A\tilde{x} = U\Sigma V^T \tilde{x} \\
U^T \tilde{y} = \Sigma V^T \tilde{x} \quad \text{Undoing the transformation by} \ U \\
\tilde{\Sigma} U^T \tilde{y} = V^T \tilde{x} \quad \text{Unscaling by} \ \tilde{\Sigma} \\
V\tilde{\Sigma} U^T \tilde{y} = x \quad \text{Undoing the transformation by} \ V
\]

Therefore, we have \( A^\dagger = V\tilde{\Sigma} U^T \), where \( \tilde{\Sigma} \) is given in part (c).

f) Use \( A^\dagger \) to solve for \( \tilde{x} \) in the following system of equations.

\[
\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \tilde{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}
\]
Solution

From the above, we have the solution given by:

\[
\tilde{x} = A^T \tilde{y} = V \Sigma U^T \tilde{y}
\]

\[
= \begin{bmatrix}
0 & 1 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3 \\
\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}
\]

Therefore, the solution to the system of equations is:

\[
\tilde{x} = \begin{bmatrix}
3 \\
\frac{1}{2} \\
-\frac{1}{2}
\end{bmatrix}
\]

4 Controllability in 2D

Consider the control of some two-dimensional linear discrete-time system

\[
\tilde{x}(k+1) = A\tilde{x}(k) + Bu(k)
\]

where \( A \) is a \( 2 \times 2 \) real matrix and \( B \) is a \( 2 \times 1 \) real vector.

a) Let \( A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \) with \( a, c, d \neq 0 \), and \( B = \begin{bmatrix} f \\ g \end{bmatrix} \). Find a \( B \) such that the system is controllable no matter what nonzero values \( a, c, d \) take on, and a \( B \) for which it is not controllable no matter what nonzero values are given for \( a, c, d \). You can use the controllability rank test, but please explain your intuition as well.

Solution

With \( B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), the system is controllable for all nonzeros \( a, c, d \), because

\[
[B, AB] = \begin{bmatrix} 1 & a \\ 0 & c \end{bmatrix}, \text{ which has full rank.}
\text{With } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ the system is not controllable because } [B, AB] = \begin{bmatrix} 0 & 0 \\ 1 & d \end{bmatrix}, \text{ which only has rank=1.}
\]

The intuition is that, due to the zero entry in \( A \), the state \( x_1 \) evolves autonomously, i.e.,

\[
\frac{dx_1(t)}{dt} = ax_1(t), \text{ hence it needs to be controlled by some input } f. \text{ On the other hand we can control } x_2 \text{ via controlling } x_1, \text{ as } \frac{dx_2(t)}{dt} = cx_1(t) + dx_2(t), \text{ which implies that } x_2 \text{ can be “tuned” by manipulating } x_1.\]
b) Let \( A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \) with \( a, d \neq 0 \). and \( B = \begin{bmatrix} f \\ g \end{bmatrix} \) with \( f, g \neq 0 \). Is this system always controllable? If not, find configurations of nonzero \( a, d, f, g \) that make the system uncontrollable.

**Solution**

No. uncontrollable when \( a = d \). In this case the matrix is just a constant \( a \) times the identity. So when you check with the controllability test, \( AB \) is just a scalar multiple of \( B \) and hence linearly dependent. The intuition is that the two states are inherently “coupled” as two eigenvalues are the same. Any control input can only move the states along a line hence the states cannot reach arbitrary points in \( \mathbb{R}^2 \).

c) We want to see if controllability is preserved under changes of coordinates. To begin with, let \( \tilde{x}(k) = V^{-1}x(k) \), please write out the system equation with respect to \( \tilde{x} \).

**Solution**

\( \tilde{x}(k) = V\tilde{z}(k) \), hence we have

\[
V\tilde{z}(k + 1) = AV\tilde{z}(k) + Bu(k)
\]

\[
\tilde{z}(k + 1) = V^{-1}AV\tilde{z}(k) + V^{-1}Bu(k)
\]

d) Now show that controllability is preserved under change of coordinates. (Hint: use the fact that \( \text{rank}(MA) = \text{rank}(A) \) for any invertible matrix \( M \)).

**Solution**

The matrix whose rank needs to be tested after the coordinate change is \([V^{-1}B, V^{-1}AVV^{-1}B] = [V^{-1}B, V^{-1}AB] = V^{-1}[B, AB] \) which has the same rank as \([B, AB] \), since \( V \) by assumption is full rank.

5 **Nonlinear circuit component**

Consider the circuit below that consists of a capacitor, inductor, and a third element with a nonlinear voltage-current characteristic:

\[
i = 2v - v^2 + 4v^3
\]
a) Write a state space model of the form

\[
\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t))
\]
\[
\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))
\]

Where \( x_1(t) = v_c(t) \) and \( x_2(t) = i_L(t) \).

**Solution**

We need to get \( \frac{dv_c}{dt} \) and \( \frac{di_L}{dt} \) in terms of \( v_c \) and \( i_L \).

All the components are in parallel, so:

\[ v_c = v_i = v \]

Using the relation of an inductor's current and voltage:

\[
v_c = L \frac{di_L}{dt}
\]
\[
\frac{di_L}{dt} = \frac{1}{L} v_c
\]

Using KCL, we can say:

\[
i_c + i_L + i = 0
\]
\[
C \frac{dv_c}{dt} + i_L + 2v - v^2 + 4v^3 = 0
\]
\[
\frac{dv_c}{dt} = \frac{1}{C} \left( -i_L - 2v_c + v_c^2 - 4v_c^3 \right)
\]

Taking equations (1) and (2) and substituting in \( x_1 \) and \( x_2 \) gives us our answer:

\[
\frac{dx_1}{dt} = f_1(x_1, x_2) = \frac{1}{C} \left( -x_2 - 2x_1 + x_1^2 - 4x_1^3 \right)
\]
\[
\frac{dx_2}{dt} = f_2(x_1, x_2) = \frac{1}{L} x_1
\]

b) Linearize the state model at the equilibrium point \( x_1 = x_2 = 0 \) and specify the resulting A matrix.
Solution

\[ A = \nabla f(\bar{\mathbf{x}}) \bigg|_{x_1=x_2=0} = \left[ \begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right] \bigg|_{x_1=x_2=0} = \left[ \begin{array}{cc} \frac{1}{C} (-2 + 2x_1 - 12x_1^2) & -\frac{1}{L} \\ \frac{1}{L} & 0 \end{array} \right] \bigg|_{x_1=x_2=0} \]

\[ A = \left[ \begin{array}{cc} -\frac{2}{C} & -\frac{1}{L} \\ \frac{1}{L} & 0 \end{array} \right] \]

c) Is the linearized system stable?

Solution

\[ \det(A - \lambda I) = \lambda^2 + \frac{2}{C} \lambda + \frac{1}{LC} = 0 \]

\[ \lambda = \frac{-\frac{2}{C} \pm \sqrt{\left(\frac{2}{C}\right)^2 - \frac{4}{LC}}}{2} \]

For a continuous system to be stable, all eigenvalues must have Re{\lambda} < 0. Since both L and C can only take positive values, the square root term will always have a real part smaller than \( \frac{2}{C} \), which means both eigenvalues will have negative real parts. The system is stable.