This homework is optional.

Solutions will be published soon, after you have had some time to try out the problems.

1 Identification of an Unknown Circuit Component

Suppose we have an unknown circuit component, which we’ll denote as $X$ and represent with the symbol $\begin{array}{c}X \\ \end{array}$. $X$ could either be a resistor, a capacitor, or an inductor, but we don’t know which one it is, nor do we know what its component value (that is, its resistance, capacitance, or inductance) could be. If you needed to identify $X$, that is figure out what kind of component $X$ is and figure out its value, you would use a tool called an RLC meter. In this problem, you will examine how an RLC meter can identify unknown circuit components with the help of transfer functions.

In circuit form, an RLC meter looks like this:

\[
\begin{array}{c}
v_{in}(t) \\ \hline \end{array} \rightarrow \begin{array}{c}X \\ \hline \end{array} \rightarrow \begin{array}{c}v_{out}(t) \\ \hline \end{array}
\]

Here, $v_{in}(t) = A_{in} \cos(2\pi f_0 t + \theta_{in})$ is a known sinusoidal test input of known frequency $f_0$, known amplitude $A_{in}$, and known phase $\theta_{in}$, while $R$ is also a known resistance. Under this setup, we know that $v_{out}(t)$ will also be a sinusoid, which we’ll denote as $v_{out}(t) = A_{out} \cos(2\pi f_0 t + \theta_{out})$.

When $X$ is connected to the RLC meter, an on-board microcontroller takes samples from $v_{in}(t)$ and $v_{out}(t)$ and uses these samples to compute $Z_X|_{f_0}$, the impedance of the unknown component at frequency $f_0$. From the value of $Z_X|_{f_0}$, it can figure out whether $X$ is a resistor, a capacitor, or an inductor, as well as what resistance, capacitance, or inductance it has.
a) Find the transfer function \( H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} \) when the unknown component is connected to the RLC meter. Here, \( \tilde{V}_{\text{out}} \) and \( \tilde{V}_{\text{in}} \) denote the phasor representations of \( v_{\text{out}}(t) \) and \( v_{\text{in}}(t) \), respectively. Answer in terms of \( R \), the known resistance, and \( Z_X(\omega) \), the unknown impedance.

**Solution**

With component \( X \) in place, the portion of the RLC meter we’ve shown you is a voltage divider, with impedances \( R \) and \( Z_X \). With that in mind, the transfer function is

\[
H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \left( \frac{R}{R + Z_X} \right) \times \left( \frac{1}{\tilde{V}_{\text{in}}} \right) = \frac{R}{R + Z_X}. \tag{1}
\]

b) Suppose that we know \( H(\omega_0) \), that is the (possibly complex) numerical value of \( H(\omega) \) at the angular frequency \( \omega_0 = 2\pi f_0 \). Show how to use the value of \( H(\omega_0) \) to calculate \( Z_X|_{f_0} \), the impedance of the unknown component at the frequency \( f_0 \). Your result should be an equation for \( Z_X|_{f_0} \) in terms of quantities whose values we know.

**Solution**

From part (a) we have an expression for \( H(\omega) \) in terms of \( Z_X \). If we take this expression at the specific angular frequency \( \omega_0 \), we get

\[
H(\omega_0) = \frac{R}{R + Z_X|_{f_0}}. \tag{2}
\]

If we solve this expression for \( Z_X|_{f_0} \), we get

\[
Z_X|_{f_0} = \frac{R}{H(\omega_0)} - R = R \left( \frac{1}{H(\omega_0)} - 1 \right). \tag{3}
\]

Since we know the value of \( R \), and we are given \( H(\omega_0) \) for this part, this equation is what we wanted to find: an equation for \( Z_X|_{f_0} \) in terms of known quantities.

c) Suppose that we knew \( Z_X|_{f_0} \). Describe how to use \( Z_X|_{f_0} \) to determine both what kind of component \( X \) is and the corresponding component value? (HINT: Physical resistances, capacitances, and inductances are always positive. And \( \frac{1}{j} = -j \) for \( j = \sqrt{-1} \).)

**Solution**

To figure out if \( X \) is a resistor, capacitor, or inductor, it would suffice to look at the phase of \( Z_X|_{f_0} \). If the phase were positive, then \( X \) would have...
to be an inductor; if it were negative, \( X \) would have to be a capacitor; and if it were zero, then \( X \) would have to be a resistor. Once we’ve decided what kind of component \( X \) is this way, we can use the magnitude of \( Z_X|_{f_0} \) to determine the value:

\[
C_X = \frac{1}{2\pi f_0 |Z_X|_{f_0}}, \quad \text{if } X \text{ is a capacitor} \tag{4}
\]

\[
L_X = \frac{|Z_X|_{f_0}}{2\pi f_0}, \quad \text{if } X \text{ is an inductor} \tag{5}
\]

\[
R_X = |Z_X|_{f_0}, \quad \text{if } X \text{ is a resistor} \tag{6}
\]

2 Circuit Discretization

Let’s consider the following RLC circuit that you have encountered before.

\[\begin{array}{c}
+ \quad C \quad I_L \quad R \quad L \\
\hline
V_C \quad + \quad V_R \quad - \\
\hline
\end{array}\]

\[u(t)\]

a) Find the matrix differential equation for the above system using the state-vector \( \vec{x} = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix} \) as

\[
\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) + \vec{b}u(t).
\]

**What is \( A \)? What is \( \vec{b} \)?**

Your answers should be in terms of \( R, L, C \).
Solution

Writing the circuit equations, we get:

\[ u(t) = V_C + V_R + V_L \]
\[ V_L = L \frac{d}{dt} I_L \]
\[ V_R = I_L R \]
\[ I_L = C \frac{d}{dt} V_C \]

Substituting the definitions, we get:

\[ u(t) = V_C + I_L R + L \frac{d}{dt} I_L \]
\[ \Rightarrow \frac{d}{dt} I_L = -\frac{1}{L} V_C - \frac{R}{L} I_L + \frac{1}{L} u(t) \]
\[ \frac{d}{dt} V_C = \frac{1}{C} I_L \]

Hence, we can write the matrix differential equation as

\[
\frac{d}{dt} \tilde{x}(t) = \begin{bmatrix}
0 & \frac{1}{L} \\
-\frac{1}{L} & -\frac{R}{L}
\end{bmatrix} \tilde{x}(t) + \begin{bmatrix}
0 \\
\frac{1}{L}
\end{bmatrix} u(t)
\]

b) Now, assume for some specific component values we get the following differential equation:

\[
\frac{d}{dt} \tilde{x}(t) = \begin{bmatrix}
0 & 1 \\
-2 & -3
\end{bmatrix} \tilde{x}(t) + \begin{bmatrix}
0 \\
2
\end{bmatrix} u(t).
\] (7)

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval \( T \). Let us discretize the above system. Assume that we use piecewise constant voltage inputs \( u(t) = u_d(k) \) for \( t \in [kT, (k+1)T) \).

Recall from the homework that for a hypothetical scalar differential equation \( \frac{dx}{dt} x(t) = \lambda x(t) + b u(t) \), we can discretize it as long as \( \lambda \neq 0 \) as follows:

\[ x_d(k+1) = e^{\lambda T} x_d(k) + \frac{e^{\lambda T}}{\lambda} b u_d(k). \] (8)

Here \( x_d(k) = x(kT) \).

Using equation (8), calculate the discrete-time system for Equation (7)'s continuous-time vector system in the form:

\[
\tilde{x}_d(k+1) = A_d \tilde{x}_d(k) + B_d u_d(k).
\]
More concretely, find $A_d$ and $\vec{b}_d$.

You do not need to multiply out any matrices. It is fine if you give your answers as explicit products of matrices/vectors/etc.

Hint: We have provided information regarding the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ in (7) for your convenience (not all of this is needed) on the opposite page.

a) The determinant of $A$: $\det(A) = 2$.

b) The trace of $A$: $\text{tr}(A) = -3$.

c) $A^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ -2 & 0 \end{bmatrix}$.

d) We can diagonalize the matrix as $A = V \Lambda V^{-1}$, where, $\Lambda$ is a diagonal matrix with the eigenvalues in its diagonal and the columns of $V$ are the eigenvectors of the corresponding eigenvalues.

e) The eigenvalues/eigenvectors for $A$ are:

For $\lambda_1 = -2$: $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

For $\lambda_2 = -1$: $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

f) For $V = [\vec{v}_1, \vec{v}_2]$, we have $V^{-1} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$.

Solution

We want to change coordinates to the eigenbasis, so that the system of differential equations looks like scalar equations. Having done so, we can discretize the problem, and then change coordinates back.

We can write $A = V \Lambda V^{-1}$, hence substituting this into our differential equation, we get:

$$\frac{d}{dt} \vec{x} = V \Lambda V^{-1} \vec{x} + \vec{b} u(t)$$

$$\frac{d}{dt} V^{-1} \vec{x} = \Lambda V^{-1} \vec{x} + V^{-1} \vec{b} u(t)$$

Writing, $\vec{z} = V^{-1} \vec{x}$, we can diagonalize the system. Hence, we can discretize the system in this diagonal space, giving us

$$\vec{z}_d(k + 1) = e^{\Lambda T} \vec{z}_d(k) + \Lambda_T \vec{b} u(t).$$

Here, $\vec{z}_d(k) = \vec{z}(kT)$ and $\Lambda_T = \text{diag}(\frac{e^{\lambda T} - 1}{\lambda})$ – for the $j$-th entry of the diagonal. Fortunately, in our case, all the eigenvalues $\lambda_j \neq 0$ and so this
applies. This is just applying the scalar solution we were given in the problem to each of the components of \( \tilde{z} \) in the discretization.

Hence, substituting back for \( \tilde{x}_d(k) \) gives

\[
V^{-1}x_d(k + 1) = e^{\Lambda^TV^{-1}\tilde{x}_d(k)} + \Lambda_TV^{-1}\tilde{u}(t)
\]

\[
x_d(k + 1) = Ve^{\Lambda^TV^{-1}\tilde{x}_d(k)} + V\Lambda_TV^{-1}\tilde{u}(t).
\]

This gives:

\[
A_d = Ve^{\Lambda^T}V^{-1}
\]

\[
\vec{b}_d = V\Lambda_TV^{-1}\vec{b}
\]

Hence we have,

\[
A_d = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{-2T} & 0 \\ 0 & e^{-T} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}
\]

\[
\vec{b}_d = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -e^{-2T} & 0 \\ 0 & -(e^{-T} - 1) \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

Multiplying out the matrices, we get:

\[
A_d = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ 2e^{-2T} - 2e^{-T} & 2e^{-2T} - e^{-T} \end{bmatrix}
\]

\[
\vec{b}_d = \begin{bmatrix} e^{-2T} - 2e^{-T} + 1 \\ 2e^{-T} - 2e^{-2T} \end{bmatrix}
\]

3 SVD stuff

a) Compute the SVD of the following matrix. Express your answer in the form of \( \sum_i \sigma_i \tilde{u}_i \tilde{v}_i^T \)

\[
A = \begin{bmatrix} \vec{a} & -\vec{a} \end{bmatrix}
\]

Here, \( \vec{a} \) is some arbitrary vector in \( \mathbb{R}^n \)

**Solution**

\[
A = \sigma_1 \tilde{u}_1 \tilde{v}_1^T
\]

where \( \tilde{u}_1 = \frac{\vec{a}}{||\vec{a}||}, \tilde{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 \sqrt{2} \end{bmatrix} \), and \( \sigma_1 = ||\vec{a}|| \times \sqrt{2} \)

b) Compute the compact form SVD of

\[
A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\]
Solution

\[ AA^T = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \]

So the eigenvalues/eigenvectors of \( AA^T \) are \( \lambda_1 = 9, \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) \( \lambda_2 = 1, \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

This means that \( \sigma_1 = 3, \sigma_2 = 1 \). To calculate \( \vec{v}_i \), we can apply the formula \( \vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i \). Thus,

\[ \vec{v}_1^T = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \end{bmatrix} \]

\[ \vec{v}_2^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \end{bmatrix} \]

4 Control Question

Given a non-linear two-dimensional system with states \( x_0 \) and \( x_1 \) and inputs \( u_0 \) and \( u_1 \) that evolves according to the following coupled differential equations:

\[ \begin{align*}
\frac{d}{dt} x_0 &= \dot{x}_0 = x_1 \\
\frac{d}{dt} x_1 &= \dot{x}_1 = \alpha - \beta \frac{u_1^2}{x_0^2} 
\end{align*} \]

where: \( \alpha, \beta > 0 \) and \( u_1 \geq 0 \)

a) Write the non-linear system (9) in a vector form \( \frac{d}{dt} \vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)) \)

Solution

The non-linear system is given in vector form as

\[ \frac{d}{dt} \vec{x}(t) = \vec{f}(\vec{x}(t), \vec{u}(t)) = \begin{bmatrix} x_1 \\ \alpha - \beta \frac{u_1^2}{x_0^2} \end{bmatrix} \]

b) Find an input vector \( \vec{u}_c \) and a state vector \( \vec{x}_c \) so that the system remains in the state vector \( \vec{x}_c = \begin{bmatrix} x_0_c \\ x_1_c \end{bmatrix} = \begin{bmatrix} 1 \\ x_1_c \end{bmatrix} \)
Solution

Setting \( \frac{d}{dt} \vec{x}_e(t) = \vec{f}(\vec{x}_e(t), \vec{u}_e(t)) = \vec{0} \) we have

\[
\begin{align*}
    x_{1e} &= 0 \\
    \alpha - \beta u_{1e} &= 0
\end{align*}
\]

Thus, we get

\[
\vec{x}_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

and

\[
\vec{u}_e = \begin{bmatrix} 0 \\ \sqrt{\frac{\beta}{\alpha}} \end{bmatrix}
\]

as input gets only positive values.

c) Write the linearized state space equations around \( \vec{x}_e \) and \( \vec{u}_e \). Convert it into the following form and find \( A \) and \( B \).

\[
\frac{d}{dt} \vec{x}(t) = A(\vec{x} - \vec{x}_e) + B(\vec{u} - \vec{u}_e)
\]

Solution

We find the Jacobian matrices for both the state and the input vectors. Thus,

\[
\frac{\partial \vec{f}}{\partial \vec{x}}|_{\vec{x}_e, \vec{u}_e} = \begin{bmatrix} 0 & 1 \\ 2\alpha & 0 \end{bmatrix} = A_e
\]

and

\[
\frac{\partial \vec{f}}{\partial \vec{u}}|_{\vec{x}_e, \vec{u}_e} = \begin{bmatrix} 0 & 0 \\ 0 & -2\sqrt{\alpha\beta} \end{bmatrix} = B_e
\]

The system evolves as

\[
\frac{d}{dt} \vec{x}(t) = A_e(\vec{x} - \vec{x}_e) + B_e(\vec{u} - \vec{u}_e)
\]

d) Prove that the linearized model is controllable for every \( \alpha, \beta > 0 \).

HINT: A system is controllable iff matrix \( C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \) is full rank and \( n \) is the number of states. In our system since \( n = 2 \) we have \( C = \begin{bmatrix} B & AB \end{bmatrix} \) full rank

8
Solution

In our case we have

\[
C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2\sqrt{\alpha\beta} \\ 0 & -2\sqrt{\alpha\beta} & 0 & 0 \end{bmatrix}
\]

which has column rank 2. Thus, the system is completely controllable for every \( \alpha, \beta \).

5 Discrete Time Control

Consider the system

\[
\begin{bmatrix} x_1(t + 1) \\ x_2(t + 1) \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)
\]

a) Determine if the system is stable.

Solution

To determine if the system is stable, we need to find the eigenvalues of \( A \).

\[
\det(\lambda I - A) = 0
\]

\[
\det(\lambda I - A) = (\lambda - 1.5)(\lambda - 0.5) - 0 \cdot 1 = (\lambda - 1.5)(\lambda - 0.5) = 0
\]

\[\lambda_1 = 0.5\]

\[\lambda_2 = 1.5\]

For a discrete system to be stable, \( |\lambda| < 1 \).

Since \( |\lambda_2| > 1 \), the system is unstable.

b) Determine the set of all \((b_1, b_2)\) values for which the system is not controllable and sketch this set of points in the \( b_1 - b_2 \) plane below.

Solution

\[
C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} b_1 & 1.5b_1 + b_2 \\ b_2 & 0.5b_2 \end{bmatrix}
\]
The system is unstable when C is not full rank. C is not full rank if:

\[
\begin{align*}
\frac{1.5b_1 + b_2}{0.5b_2} &= \frac{b_1}{b_2} \\
b_2(1.5b_1 + b_2) - 0.5b_1b_2 &= 0 \\
b_2(b_1 + b_2) &= 0
\end{align*}
\]

The system is not controllable if \(b_2 = 0\) or \(b_1 + b_2 = 0\).

6 PCA Midterm question

Consider a matrix \(A \in \mathbb{R}^{2500 \times 4}\) which represents the EE16B Sp’2025 midterm 1, midterm 2, final and lab grades for all 2500 students taking the class.
To perform PCA, you subtract the mean of each column and store the results in $\tilde{A}$. Your analysis includes:

a) Computing the SVD: 
\[ \tilde{A} = \sigma_1 \tilde{u}_1 \tilde{v}_1^T + \sigma_2 \tilde{u}_2 \tilde{v}_2^T + \sigma_3 \tilde{u}_3 \tilde{v}_3^T + \sigma_4 \tilde{u}_4 \tilde{v}_4^T \]
and plot the singular values.

b) Computing the graph $\tilde{u}_1^T \tilde{A}$ and $\tilde{u}_2^T \tilde{A}$

The analysis data are plotted below:

Based on the analysis, answer the following true or false questions. Briefly explain your answer.

a) The data can be approximated well by two principle components.

**Solution**

True. From graph (i), there are two significant singular values, the rest are small.

b) The students’ exam scores have significant correlation between the exams.
Solution

True. \( u_i^T \tilde{a}_i \) gives the correlation between the \( i \)th principle component and the variable stored in the \( i \)th column of \( A \). From graph (ii), both midterms as well as final grades are highly correlated with the first principle component while all three also have little to no correlation to the second principle component. This means the exam scores are highly correlated with each other.

c) The middle plot (ii) shows that students who did well on the exam did not do well in the labs and vice versa.

Solution

False. Graph (ii) shows that the exam scores and lab scores are not correlated at all. i.e. we cannot determine a student’s lab score from their exam scores and vice versa.

d) One of the principle components attributes is solely associated with lab scores and not with exam scores.

Solution

True. From graph (ii), the second principle component is only highly correlated with the lab scores.