Calculating the Singular Value Decomposition

Suppose we have a matrix \( A \) of dimension \( m \times n \) \((n > m)\) with rank \( r \).

We can find the singular value decomposition (SVD)

\[
A = \sum_{i=1}^{r} \sigma_i \tilde{u}_i \tilde{v}_i^T
\]

with the following steps.

1. Find the eigenvalues \( \lambda_i \) of \( A^T A \) and order them such that \( \lambda_1 \geq \ldots \geq \lambda_r > 0 \) and \( \lambda_{r+1} = \cdots = \lambda_n = 0 \).

2. Find the orthonormal eigenvectors of \( A^T A \), so that

\[
A^T A \tilde{v}_i = \lambda_i \tilde{v}_i, \quad i = 1, \ldots, r
\]

Note that the vectors must be orthonormal, that is \( \tilde{v}_i^T \tilde{v}_i = 1 \) and \( \tilde{v}_i^T \tilde{v}_j = 0 \) for \( i \neq j \).

3. Let \( \sigma_i = \sqrt{\lambda_i} \) and set

\[
\tilde{u}_i = \frac{A \tilde{v}_i}{\sigma_i}, \quad i = 1, \ldots, r
\]

Note: We will see later that real symmetric matrices \( Q = Q^T \) have real eigenvalues and a set of real, orthonormal eigenvectors. Moreover if we can write \( Q = R^T R \), the eigenvalues are non-negative.
1 SVD Example

Define the matrix

\[ A = \begin{bmatrix}
1 & -1 \\
-2 & 2 \\
2 & -2
\end{bmatrix}. \]

a) Find the SVD of \( A \).

b) Find the rank of \( A \).
c) Find a basis for the nullspace of $A$.

d) Find a basis for the range (or columnspace) of $A$. 

e) Create the SVD of $A^T$. What are the relationships between the answers to (a)-(d) for $A$ and for $A^T$?