Continuous and discrete time

There are two different dialects for modeling change over time. Thus far we have modeled real-life events using differential equations and initial conditions. For example, the voltage across a capacitor connected to a voltage source by a resistor is fully described by the following differential equation and initial conditions.

\[
\frac{d}{dt} v_C(t) = -\frac{1}{RC} v_C(t) + \frac{1}{RC} v_{in}(t), \quad v_C(0) = v_0
\]  

(1)

Abstracting away particulars, continuous-time scalar linear systems can be represented in variants of the following form:

\[
\frac{d}{dt} x(t) = \lambda x(t) + \mu u(t), \quad x(0) = x_0.
\]  

(2)

This discussion will introduce discrete-time scalar linear systems, which have models similar to the following:

\[
x[t + 1] = a x[t] + b u[t], \quad x[0] = x_0.
\]  

(3)

Notice that evolution is represented by defining the transition from \( x[t] \) to \( x[t + 1] \). The state \( x \) is not a continuous function of time, but a sequence of individual moments. Can you think of systems in life that are naturally more susceptible to discrete-time modeling?

1 Differential equations with piecewise constant inputs

1. Let \( x(\cdot) \) be a solution to the following differential equation:

\[
\frac{d}{dt} x(t) = \lambda (x(t) - u(t)).
\]  

(4)

Let \( T > 0 \). Let \( x[\cdot] \) “sample” \( x(\cdot) \) as follows:

\[
x[n] = x(nT).
\]  

(5)

Assume that \( u(\cdot) \) is constant between samples of \( x(\cdot) \), i.e.

\[
u(t) = u[n] \quad \text{when} \quad nT \leq t < (n + 1)T.
\]  

(6)

For a general time-step \( n \), write \( x[n + 1] \) in terms of \( x[n] \) and \( u[n] \). Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear system.

Answer

As \( u = u[n] \) is constant between samples \( x[n] \) and \( x[n + 1] \), the following differential equation and initial conditions describe what is happening to \( x \) during this interval:

\[
\frac{d}{dt} x(t) = \lambda (x(t) - u[n]), \quad x(nT) = x[n].
\]  

(7)
Let’s use the entire RHS of the differential equation as a change of variables from $x$ to $z$. (Other changes of variables are possible, e.g. $z_{\text{alt}} = x(t) - u(t)$.)

$$z(t) = \lambda(x(t) - u[n]) \quad (8)$$

By differentiating both sides of this relationship, we can achieve a differential equation for $z$.

$$\frac{d}{dt} z(t) = \frac{d}{dt} (\lambda(x(t) - u[n])) \quad (9)$$

$$= \lambda \frac{d}{dt} x(t) - \lambda \frac{d}{dt} u[n] \quad (10)$$

As $u[n]$ is a constant, $\frac{d}{dt} u[n] = 0$.

$$= \lambda \frac{d}{dt} x(t) \quad (11)$$

Apply the differential equation for $x$.

$$= \lambda \left( \lambda(x(t) - u[n]) \right) \quad (12)$$

We can recognize the expression in the parentheses as $z(t)$.

$$\frac{d}{dt} z(t) = \lambda z(t) \quad (13)$$

A solution to this equation is of the form $z(t) = Ke^{\lambda t}$, where $K$ is a constant to be determined. By reversing Eqn. 8, we arrive at a solution for $x(t)$—where, still, $K$ remains to be determined.

$$x(t) = \frac{1}{\lambda} z(t) + u[n] \quad (14)$$

$$= \frac{1}{\lambda} Ke^{\lambda t} + u[n] \quad (15)$$

We will determine $K$ by insisting that our solution comply with the initial conditions of Eqn. 7.

$$x(nT) = \frac{1}{\lambda} Ke^{\lambda(nT)} + u[n] = x[n] \quad (16)$$

$$K = \frac{\lambda}{e^{\lambda(nT)}} (x[n] - u[n]) \quad (17)$$

Now we have enough to evaluate $x((n + 1)T)$, by evaluating $x(t)$ at $(n + 1)T$.

$$x((n + 1)T) = \frac{1}{\lambda} \left( Ke^{\lambda(n+1)T} + u[n] \right) \quad (18)$$

$$= \frac{1}{\lambda} \left( \frac{\lambda}{e^{\lambda(nT)}} (x[n] - u[n]) e^{\lambda(n+1)T} \right) + u[n] \quad (19)$$

$$= e^{\lambda T} x[n] + \left( 1 - e^{\lambda T} \right) u[n] \quad (20)$$
Rewrite \( x((n+1)T) \) as \( x[n+1] \):

\[
x[n+1] = e^{\lambda T}x[n] + \left(1 - e^{\lambda T}\right)u[n]
\]  

(21)

and we are done.

2. Let \( T = 1 \) and \( \lambda = -100 \). Sketch a piecewise constant input \( u[\cdot] \) of your choice, then sketch \( x(t) \). Mark \( x[n] \). Your sketch doesn’t have to be exact, but you should be able to supply analysis to justify why it looks a certain way: how are you using the fact that \( \lambda T \) is large and negative?

**Answer**

A typical drawing might look similar to this:

![Diagram](image)

Notice that the displacement between \( x(t) \) and its moving target \( u(t) \) is always in exponential decay (it is proportional to \( z(t) \)). Because \( \lambda T \) is large and negative, \( e^{\lambda T} \approx 0 \), so

\[
x[n+1] = e^{\lambda T}x[n] + \left(1 - e^{\lambda T}\right)u[n]
\]  

(22)

\[
x[n+1] \approx u[n]
\]  

(23)

3. Let \( T = 1 \) and \( \lambda = -1 \). Define \( u[n] \) as follows:

\[
u[n] = \begin{cases} 
1, & \text{if } n \text{ is even} \\ 
-1, & \text{if } n \text{ is odd} 
\end{cases}
\]  

(24)
Answer

Notice how $u$, which is $x$’s target, is flipping so quickly that $x$ never gets close to the finish line. It gets partway there and then is told to turn around. An approximate sketch (with features exaggerated) would look like this: