D) LINEARITY (FOR SCALAR FUNCTIONS)

→ GIVEN \( y = f(x) \)
→ PICTORIALLY:

\[
\begin{align*}
\text{LINEAR} & \quad & \text{NONLINEAR}
\end{align*}
\]

→ ALGEBRAICALLY: \( f(x) \) is LINEAR if and only if (iff) it satisfies both:

→ SCALING: Given any \( x \), and any \( a \), \( f(ax) = a f(x) \)
→ SUPERPOSITION: Given any \( x \) and any \( y \), \( f(x+y) = f(x) + f(y) \)

→ EXAMPLE: WHY IS \( f(x) \) NOT LINEAR?

→ EXAMPLE: WHY IS \( f(x) \) LINEAR?

→ NOTE: To show that \( f(x) \) is NONLINEAR, you need find only one \( x \) and one \( x/y \) where scaling or superposition does not work. But to show that it is LINEAR, you have to prove it for every \( x/x/y \). This is usually only possible algebraically.
2) **Linearization of a Scalar Function**

- **Given (Non-Linear) $f(x)$**
- **And some specific value of $x$, call it $x^*$ ("Expansion Point")**
- **The "linearization of $f(x)$ about $x^*$ means:**
  - The straight line going through $(x^*, f(x^*))$, with slope $\frac{df}{dx} \bigg|_{x^*}$
  - i.e., The tangent line at $x^*$

- **Algebraically:**
  \[
  f_L(x) = f(x^*) + m \cdot \Delta x
  \]
  - $m = \frac{df}{dx} \bigg|_{x^*}$ = slope of $x^*$
  - $\Delta y = f_L(x^*+\Delta x) - f(x^*) = m \cdot \Delta x$

- **Tangent line through $(x^*, f(x^*))$**

- **Note that $\Delta y$ is a linear function of $\Delta x$ (hence the term linearization)**

- **Why is linearization relevant?**
  - Because $f_L(x)$ approximates $f(x)$, and the approximation gets better and better as $x$ gets closer to $x^*$ ($\Delta x$ becomes smaller and smaller)

- **More precisely:**
  \[
  \frac{f(x) - f_L(x)}{x - x^*} \to 0 \text{ as } x \to x^* \]
  - Details: Dis + HW

- **Intuitively:** As you blow up $f(x)$ around $x^*$, it looks more and more like its tangent line, which is exactly what $f_L(x)$ is.
3) **LINEARITY / NONLINEARITY FOR A SYSTEM (WITH I/O)**

\[ u(t) \xrightarrow{\text{SYSTEM}} y(t) \]

**FUNCTIONAL**

\[ F = \sum_{i=1}^{n} \left( \int_{0}^{t_i} u(t) \, dt \right) \]

**FUNCTIONAL:**

\[ g(t) = \sum_{i=1}^{n} u(t) \]

**YOU NEED THE ENTIRE WAVEFORM FOR** \( u(t) \), \( \forall t \), **BEFORE YOU CAN**

**EXAMPLE**

\[ y(t) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} u(t) \, dt \]

\( y(t) \) **IS THIS AREA**

**IT DEPENDS ON ALL THE VALUES OF** \( u(t) \) **BETWEEN** \( t=3 \) **AND** \( t=4 \) **MEMORY**
LINEARITY FOR SYSTEMS:

A SYSTEM IS LINEAR IFF IT SATISFIES BOTH SCALING AND SUPERPOSITION:

Scaling: \( \mathcal{L}\{\alpha u(t)\} = \alpha \mathcal{L}\{u(t)\} \) \( \forall \alpha \), \( \forall u(t) \)

NOTE: \( \alpha \) is a CONSTANT (NOT a function of time)

Superposition: \( \mathcal{L}\{u_1(t) + u_2(t)\} = \mathcal{L}\{u_1(t)\} + \mathcal{L}\{u_2(t)\} \) \( \forall u_1(t), u_2(t) \)

Now you see that linearity for \( f(x) \) was just a simple special case of this.

Graphically (must hold for every choice of \( u(t), 0, u_1(t), u_2(t) \))

Example: Given \( \frac{dx}{dt} = f(x) + bu(t) \); all scalars. \( u(t) = \) the input, \( x(t) = \) the output.

Suppose \( f(x) \) is linear. Is the ODE system linear?

Test scaling: Is it true that for any \( u(t) \), any \( \alpha \), if you input \( \alpha u(t) \), then you get \( \alpha x(t) = \) output, when \( x(t) \) is the output if you input \( u(t) \)?

I.e., does \( (\alpha u(t), \alpha x(t)) \) satisfy the ODE?

Suppose \( y(t) \) is the output if you input \( \alpha u(t) \): \( \frac{dy}{dt} = \left(\frac{dy(t)}{dt}\right) = f(y(t)) + bu(t) \)

Is \( y(t) = \alpha x(t) \)? Try it: \( \frac{d}{dt} (\alpha x(t)) = \alpha \frac{d}{dt} x(t) = \alpha (f(x(t)) + bu(t)) \)

Since \( f(x) \) is linear, \( \alpha f(x(t)) = f(\alpha x(t)) \) \( \Rightarrow \frac{d}{dt} (\alpha x(t)) = f(\alpha x(t)) + b(\alpha u(t)) \)
4) **LINEARIZATION OF AN ODE SYSTEM (SCALAR) ABOUT AN OPERATING PT**

- **GIVEN** \( \frac{dx}{dt} = f(x) + bu(t) \), \( f(x) \) is nonlin.
- **STEP 1**: CHOOSE A DC INPUT \( u(k) = u^* \), CONSTANT WITH TIME.
- **STEP 2**: SOLVE THE ODE \( \frac{dx}{dt} \) ON DC SOLUTION \( x(k) = x^* \), I.E., SOLVE

\[
\frac{dx^*}{dt} = -6u^*
\]

**TO FIND** \( x^* \).

\( x^* \) = equilibrium point, also (DC) operating point, also fixed point.

**STEP 3**: DEFINE \( Sx(t) \in x(t) - x^* \), \( Su(t) = u(t) - u^* \) and re-express the ODE (4-2)

\[
\frac{d}{dt} [x^* + Sx(t)] = f(x^* + Sx(t)) + b(u^* + Su(t))
\]

\[
\Rightarrow \frac{d}{dt} Sx(t) = f(x^* + Sx(t)) + 6u^* + bSu(t)
\]

**NOTE**: \( Su(t) \) is often called the **INPUT PERTURBATION** (about \( u^* \)), and \( Sx(t) \) the **OUTPUT DEVIATION** (about \( x^* \)).
→ Step 4: Suppose you are given that $\delta u(t)$ is "small":

$u(t)$ is the input; you might be able to arrange for $u(t)$ to stay near $u^*$. 

$u(t) = u^* + \delta u(t)$: "stays near $u^*$";

$\delta u(t)$ is "small".

→ Step 5: Assume that $\delta x(t)$ (which the system produces by itself, in response to $\delta u(t)$) is also small. Very big assumption. It is possible for it not to be true.

→ Step 6: Under this assumption (i.e., that $\delta x(t)$ is small enough), linearizing $f(x)$ as

$f(x^* + \delta x(t)) \approx f(x^*) + \frac{df}{dx}igg|_{x^*}(\delta x)$

is a reasonable approximation. Use it in (4-2).

Note: $\delta x(t)$ "small" means small enough that (4-3) is a good approximation.
Step 6 (Continued): Putting (4.3) in (4.2)

\[ \frac{d}{dt} s_t(t) = f(x^* + s_t(t)) + b u^* + b s_t(t) \]

\[ = f(x^*) + m s_t(t) + b u^* + b s_t(t) \]

\[ \Rightarrow \text{Since } x^* \text{ is the operating point, defined by (4.1), we have: } f(x^*) + b u^* = 0 \]

\[ \Rightarrow \frac{d}{dt} s_t(t) = m s_t(t) + b s_t(t) \]

(4-4)

This is called the linearization of (4.5) about its op. pt. \( x^* \) (and input \( u^* \)).

\[ \text{It is a linear system with input } s_t(t) \text{ and output } s_t(t) \]

\[ \text{It is typically much easier to solve than (4.5), which is nonlinear.} \]

\[ \text{Solving it can validate (or invalidate) the assumption in Step 5, that } s_t(t) \text{ is small} \]

\[ \text{Discussion} \]

When valid, it can provide tremendous insight into the behavior of the original nonlinear ODE.
Example: \[
\frac{dx}{dt} = f(x) + u(t), \quad \text{where } f(x) = \cos(x) \]

- choose \( u^* = 0 \) \( \Rightarrow \) \( \cos(x^*) = 0 \) \( \Rightarrow x^* = \frac{\pi}{2} \)
  \( \vee \) \( 3\frac{\pi}{2}, -\frac{\pi}{2}, +\infty \).

- i.e., for the same DC input \( u^* \), this system has multiple operating points!

Suppose we take \( x^* = \frac{\pi}{2} \), then \( f(x^* + \delta x) \approx -\delta x \), if \( \delta x \) is small.

- System Linearization:
  \[
  \frac{d}{dt} \delta x(t) = -f(x(t) + \delta x(t))
  \]

- By solving this system, it can be shown that if \( \delta u(t) \) is small, \( \delta x(t) \) will also be correspondingly small.

- So STEP 5's assumption (\( \delta x(t) \) small) is indeed valid!

But if you take \( x^* = 3\frac{\pi}{2} \), then the linearization becomes:

\[
\frac{d}{dt} \delta x(t) = +f(x(t) + \delta x(t))
\]
for which the solution blows up unboundedly, in general, even if \( \delta u(t) \) is small.

So STEP 5's assumption (\( \delta x(t) \) small) is invalid, and subsequent steps are wrong! i.e., STEP 5's assumption led to a contradiction.

We should not use the linearization; it will not approximate the original nonlinear system well, even for small \( \delta u(t) \).
- except during short periods of time when the assumption might hold!

\[ u(t) \]

\[ u^* + \delta u(t) \]

\[ \epsilon \]

\[ \delta u(t) \]

\[ x^* \]

\[ \epsilon \]

\[ \delta x(t) \]

\[ \delta x(t) \text{ BLOWS UP} \]

\[ \text{NOT SMALL} \]

STEP 4

STEP 5
\[ \frac{dx}{dt} = f(x) + u(t), \quad \text{where} \quad f(x) = x^3 \]

\[ \Rightarrow \text{Choose} \quad u^* : 1 \Rightarrow x^{**} + 1 = 0 \Rightarrow x^* = -1 \]

\[ \Rightarrow \frac{df}{dx} \bigg|_{x^*} = 3x^2 = 3 = \gamma_0 \]

\[ \Rightarrow f(x^* + \delta x) \approx -1 + 3\delta x \]

\[ \Rightarrow f(x^*) \]

\[ \text{SYSTEM LINEARIZATION:} \]

\[ \frac{dx(t)}{dt} = 3x(t) + \delta u(t) \]

\[ \Rightarrow \text{SOLUTION:} \quad \text{even with no input perturbation (} \delta u(t) = 0), \text{ but with any non-zero IC } x(0), \text{ the solution increases without bound!} \]

\[ \Rightarrow x(t) = x(0) e^{3t} \]

\[ \Rightarrow x(t) \text{ becomes arbitrarily large} \Rightarrow \text{STEP 5 ASSUMPTION (INVALID!)} \]