1 Periodic Waveforms

Periodic waveforms are signals $x(t)$ that repeat the same pattern in a set amount of time $T$, which is called the period of $x(t)$. Mathematically, we say $x(t)$ is $T$-periodic if

$$x(t + T) = x(t)$$

for all time $t$. The fundamental frequency $f_0$ of the signal is given by

$$f_0 = \frac{1}{T}.$$ 

One periodic signal we’ve used many times is the sinusoid. The natural period of a sinusoid is $2\pi$. We define a cosine with period $T$ by

$$x(t) = \cos \left( \frac{2\pi}{T} t \right).$$

Periodic waveforms are extremely useful in many engineering contexts. An obvious example is alternating current (AC), which is a pure sinusoid. But often we use other periodic functions besides sinusoids. For example, pulse width modulation (PWM) waveforms have the form

$$p(t) = \begin{cases} 1 & \text{if } 0 \leq t < \alpha T \\ \alpha & \text{if } \alpha T \leq t < T \\ -1 & \text{if } T \leq t < 2T \end{cases}$$

where $\alpha \in [0, 1]$ defines the duty-cycle of the signal.

Pulse width modulation is used to change the brightness of LED’s, as well as in DC-DC power conversion. We can produce a PWM waveform $p(t)$ by passing a $T$-periodic sine $y(t)$ with an amplitude (DC) and phase offset through a nonlinear function $f(\cdot)$:
Here, \( p(t) = f(y(t)) \). As we can see above, applying this nonlinear function to a \( T \)-periodic signal results in a \( T \)-periodic waveform. This is in fact a more general property: a function applied to a \( T \)-periodic sinusoid will produce a (more complicated) \( T \)-periodic waveform.

Therefore, we might consider that an arbitrary \( T \)-periodic waveform is related to an underlying \( T \)-periodic sinusoid.

2 Fourier Series

The Fourier Series says we can represent any \( T \)-periodic waveform as the sum of many sine waves with frequencies at integer multiples of the fundamental frequency \( f_0 \):

\[
f = 0, f_0, 2f_0, 3f_0, \ldots
\]

where \( f = 0 \) is the DC component, \( f = f_0 \) is the fundamental frequency, and \( f = 2f_0, 3f_0, \ldots \) are the harmonics.

Mathematically, we can write any \( T \)-periodic waveform \( x(t) \) as

\[
x(t) = \sum_{i=0}^{\infty} B_i \cos(2\pi f_0 t + \theta_i) = \sum_{l=-\infty}^{\infty} A_l e^{j2\pi f_0 t}, \tag{1}
\]

where \( B_i \) is a real-valued amplitude, \( \theta_i \) is a real-valued phase offset, and \( A_l \) is a complex-valued coefficient. This is called the Fourier Series representation of the signal. We calculate the coefficients \( A_l \) as

\[
A_l = \frac{1}{T} \int_{0}^{T} e^{-j2\pi f_0 t} x(t) dt \tag{2}
\]

for each integer \( l \in [-\infty, \infty] \).

We can see that the Fourier Series represents a waveform with an infinite number of sinusoids up to infinitely high frequencies. But what if we only want to represent a waveform with a finite number of sinusoids?

Let us truncate the Fourier Series representation of \( x(t) \) to a summation of \( N = 2M + 1 \) sinusoids and use the fact that \( f_0 = \frac{1}{T} \):

\[
x(t) = \sum_{i=-M}^{M} X_i e^{j2\pi \frac{t}{T}}. \tag{3}
\]

This format should remind you of the Discrete Fourier Transform, where we are representing a discrete waveform as a summation of discrete sinusoids - except, of course, that the Fourier Series is a representation of continuous waveforms, rather than discrete waveforms. We will explore this connection further below.
3 Questions

1. Periodic Waveforms

Are the following functions $x(t)$ periodic? If so, what is the function’s period $T$ and fundamental frequency $f_0$?

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>Periodic?</th>
<th>$T$</th>
<th>$f_0$</th>
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<tbody>
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<td>$e^{jt}$</td>
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<td>$C$ where $C$ is a constant</td>
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2. Fourier Series and the DFT

In this problem, you will discover how we can use the DFT of $N$ samples of a continuous waveform $x(t)$ to calculate the truncated Fourier Series representation of $x(t)$.

Let’s explore the connection between the truncated Fourier Series and the DFT further. As a reminder, the truncated Fourier Series has us represent continuous, $T$-periodic signal $x(t)$ as

$$
 x(t) = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{T}it}.
$$

(a) We take $N = 2M + 1$ samples of the $T$-periodic function $x(t)$ across a single period $T$. That is, our samples are at

$$
 t = \frac{T}{N} k, \; k \in \{0, 1, 2, \cdots, N - 1\}.
$$

How can we represent the $k^{th}$ sample of $x(t)$, $x_k$?

(b) Write the relationship between the $x_k$ (samples) and $X_i$ (Fourier Series coefficients) in matrix-vector form. That is, given

$$
 \begin{bmatrix}
 x_0 \\
 x_1 \\
 x_2 \\
 \vdots \\
 x_{N-1}
 \end{bmatrix}
 = A
 \begin{bmatrix}
 X_{-M} \\
 X_{-M+1} \\
 \vdots \\
 X_{M-1} \\
 X_M
 \end{bmatrix},
$$

what is the matrix $A$? What is this matrix’s relationship to the DFT matrix, $F_N$?
(c) If we reorder the $X_i$ Fourier Series coefficients to be in "DFT order" (that is, in frequency order $f = 0, f_0, 2f_0, \cdots, Mf_0, -Mf_0, \cdots, -2f_0, -f_0$) show that

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix} = F_N^* \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_M \\ X_{-M} \\ \vdots \\ X_{-1} \end{bmatrix}. \quad (6)$$

3. Fourier Series

Suppose we have the $T$-periodic waveform $x(t)$:

That is, on the interval $[0, T)$, $x(t)$ is given by

$$x(t) = \begin{cases} 2, & 0 \leq t < \frac{3}{4}T \\ -2, & \frac{3}{4}T \leq t < T \end{cases}$$

and is $T$-periodic outside $t = [0, T)$.

(a) Calculate the Fourier Series coefficients $A_l$, given by

$$A_l = \frac{1}{T} \int_{0}^{T} e^{-j\frac{2\pi}{T}lt} x(t) dt$$

for integer $l \in [-\infty, \infty]$.

Contributors:

- Regina Eckert.