Discrete Fourier Transform - Signal Analysis

In this discussion, we want to analyze a signal of the following form:

\[ x(t) = A_0 + \sum_{l=1}^{M} A_l \cos(2\pi(lf_0)t + \theta_l) \]

where \( f_0 \) is a known frequency and \( A_0 \in \mathbb{R}, \theta_l \in \mathbb{R} \) and \( A_l \geq 0 \) for \( l = 1, \ldots, M \). By analyzing this signal, what we mean is that we want to identify \( A_l \) for \( l = 0, \ldots, N-1 \) and \( \theta_l \) for \( l = 1, \ldots, N-1 \). We will also define for \( l = 1, \ldots, N-1 \)

\[ x_l : t \mapsto A_l \cos(2\pi(lf_0)t + \theta_l) \]

and \( x_0 : t \mapsto A_0 \), the DC component of our signal.

The aim of this discussion is to show that given well-chosen samples of this signal, we can get what we want by simply computing the DFT of the sample vector \( \vec{x} \). We will denote the DFT matrix for \( N \) samples by \( F_N \), throughout this discussion.

To show this, we will go through three parts:

• Going through the proof of the DFT matrix’s orthogonality, which is a particularly useful property here

• Showing that the coefficients of \( \vec{X}_l = F_N \vec{x}_l \) have distinct non zero components.

• Finally computing \( A_l, \theta_l \) for each \( l = 0, \ldots, M \).

1. Orthogonality of the DFT matrix

(a) Let \( N \geq 2 \). We want to prove the orthogonality of the rows of the DFT matrix, \( F_N \). Let \( \omega_N = e^{j2\pi/N} \).

Remember that the \((i,k)^{th}\) entry of \( F_N \) is \( \omega_{N}^{-ik} \).

\[
F_N = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega_N^{-1} & \omega_N^{-2} & \ldots & \omega_N^{-(N-1)} \\
1 & \omega_N^{-2} & \omega_N^{-4} & \ldots & \omega_N^{-(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega_N^{-(N-1)} & \omega_N^{-2(N-1)} & \ldots & \omega_N^{-(N-1)(N-1)}
\end{pmatrix} = \begin{pmatrix}
\vec{u}_0^\top \\
\vdots \\
\vec{u}_{N-1}^\top
\end{pmatrix}
\] (1)

(b) Show that for any complex number \( z \), \( z^n - 1 = (z - 1)(1 + z + \cdots + z^{n-1}) \).
(c) Deduce from this that if \( z \) is such that \( z^N = 1 \) but \( z \neq 1 \), then \( 1 + z + \cdots + z^{N-1} = 0 \).

(d) Compute \( \vec{u}_i^T \vec{u}_k \), for any pair \((i, k)\).

2. Discrete Fourier Transform of Harmonics

In this part, we suppose that our signal is simply the \( l \)-th harmonic:

\[
x(t) = x_l(t) = A_l \cos\left(2\pi(l f_0) t + \theta_l\right)
\]

i.e. it is a sinusoidal signal of frequency \( l f_0 \), but all we know about \( l \) is that it is in \( 1, \ldots, M \).

**The task is to find \( A_l, \theta_l \) and \( l \) at the same time.** It turns out that the Discrete Fourier Transform allows us to solve this problem by taking \( N = 2 * M + 1 \) samples of this signal, at time points \( \{0, \Delta, 2\Delta, \ldots, (N-1)\Delta\} \), where \( \Delta = \frac{T}{N} \) and \( T \) is the smallest time period over which, all of the above signals are periodic. We suppose in the following that \( l \geq 1 \).

(a) **What is the smallest value of \( T \)** so that every cosine signal having a frequency of \( \{f_0, 2f_0, \ldots, Mf_0\} \) would be periodic?

(b) Before we move forward, let’s remind ourselves of Euler’s formula. **Write down cosine(x) in terms of \( e^{ix} \) and \( e^{-ix} \).**

(c) Let’s consider the vector of samples of \( x_l(t) \) at time \( k\Delta \) denoted as \( \vec{x}_l \). **Compute the expression for the components of \( \vec{x}_l \) using the previous expression for cosine and using \( \omega = e^{j\frac{2\pi}{N}} \).** Finally write \( \vec{x}_l \) using the vectors \( \vec{u}_l \), where \( \vec{u}_l^T = \begin{pmatrix} 1 & \omega_l^{-1} & \omega_l^{-2i} & \cdots & \omega_l^{-(N-1)i} \end{pmatrix} \).

(d) **Compute the Discrete Fourier Transform of \( \vec{x}_l \), i.e. \( \vec{X}_l = F_N \vec{x}_l \).** Start by writing \( F_N \) out using the \( \vec{u}_k \) vectors for \( k = 0, \ldots, N-1 \). **Show that this allows to identify \( l, A_l \) and \( \theta_l \).**

3. Analyzing a harmonic signal

Let us go back to our original problem, analyzing \( x(t) = A_0 + \sum_{l=1}^{M} A_l \cos\left(2\pi(l f_0) t + \theta_l\right) \).

(a) Show how to compute \( A_0 \).

(b) Show how to identify the quantities \( A_l \) and \( \theta_l \) for each \( l \).

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