1. Changing behavior through feedback

In this question, we discuss how feedback control can be used to change the effective behavior of a system.

(a) Consider the scalar system:

\[ x(i + 1) = 0.9x(i) + u(i) + w(i) \]  

where \( u(i) \) is the control input we get to apply based on the current state and \( w(i) \) is the external disturbance.

Is the system stable? If \( |w(i)| \leq \epsilon \), what can you say about \( |x(i)| \) at all time if you further assume that \( u(i) = 0 \) and the initial condition \( x(0) = 0 \)? How big can \( |x(i)| \) get?

(b) Suppose that we decide to choose a control law \( u(i) = kx(i) \) to apply in feedback. For what values of \( \lambda \) can you get the system to behave like:

\[ x(i + 1) = \lambda x(i) + w(i) \]  

vis-a-vis the disturbance \( w(i) \)? How would you pick \( k \)?

(c) For the previous part, which \( k \) would you choose to minimize how big \( |x(i)| \) can get?

(d) What if instead of a 0.9, we had a 3 in the original equation? What, if anything, would change?

(e) Now suppose that we have a vector-valued system with a vector-valued control:

\[ \vec{x}(i + 1) = A\vec{x}(i) + B\vec{u}(i) + \vec{w}(i) \]  

where we further assume that \( B \) is an invertible square matrix.

Suppose we decide to apply linear feedback control using a square matrix \( K \) so we choose \( \vec{u}(i) = K\vec{x}(i) \).

For what values of matrix \( G \) can you get the system to behave like:

\[ \vec{x}(i + 1) = G\vec{x}(i) + \vec{w}(i) \]  

vis-a-vis the disturbance \( \vec{w}(i) \)? How would you pick \( K \) given knowledge of \( A, B \) and the desired goal dynamics \( G \)?

2. Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

\[ \vec{x}(t + 1) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \vec{w}(t) \]
(a) Is this system controllable from $u(t)$?

**Answer:** We calculate

$$\mathcal{R}_2 = [AB, B] = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Observe that $\mathcal{R}_2$ matrix is full rank and hence our system is controllable.

(b) Is the linear discrete time system stable?

**Answer:** We have to calculate the eigenvalues of matrix $A$. Thus,

$$\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2$$

Since the magnitude of the eigenvalue $\lambda_2$ is greater than 1, the system is unstable.

(c) Derive a state space representation of the resulting closed loop system using state feedback of the form $u(t) = [k_1, k_2] \vec{x}(t)$

**Answer:** The closed loop system using state feedback has the form

$$\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot ([k_1, k_2] \vec{x}(t))$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot [k_1, k_2] \vec{x}(t)$$

Thus, the closed loop system has the form

$$\vec{x}(t+1) = \begin{bmatrix} k_1 & 1 + k_2 \\ 2 & -1 \end{bmatrix} \vec{x}(t)$$

(d) Find the appropriate state feedback constants, $k_1, k_2$ in order the state space representation of the resulting closed loop system to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$

**Answer:** $k_1 = 1, k_2 = -\frac{11}{8}$

(e) Is the system now stable?

**Answer:** Yes

(f) Suppose that instead of $\begin{bmatrix} 1 & 0 \end{bmatrix} u(t)$ in (5), we had $\begin{bmatrix} 1 & 1 \end{bmatrix} u(t)$ as the way that the discrete-time control acted on the system. Is this system controllable from $u(t)$?

(g) For the part above, suppose we used $[k_1, k_2]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

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