1 Periodic Waveforms

Periodic waveforms are signals $x(t)$ that repeat the same pattern in a set amount of time $T$, which is called the period of $x(t)$. Mathematically, we say $x(t)$ is $T$-periodic if

$$x(t + T) = x(t)$$

for all time $t$. The fundamental frequency $f_0$ of the signal is given by

$$f_0 = \frac{1}{T}.$$ 

One periodic signal we’ve used many times is the sinusoid. The natural period of a sinusoid is $2\pi$. We define a cosine with period $T$ by

$$x(t) = \cos\left(\frac{2\pi}{T}t\right).$$

Periodic waveforms are extremely useful in many engineering contexts. An obvious example is alternating current (AC), which is a pure sinusoid. But often we use other periodic functions besides sinusoids. For example, pulse width modulation (PWM) waveforms have the form

$$p(t) = \begin{cases} 1 & \text{if } 0 \leq t < \alpha T \\ -1 & \text{if } \alpha T \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in [0, 1]$ defines the duty-cycle of the signal.

Pulse width modulation is used to change the brightness of LED’s, as well as in DC-DC power conversion. We can produce a PWM waveform $p(t)$ by passing a $T$-periodic sine $y(t)$ with an amplitude (DC) and phase offset through a nonlinear function $f(\cdot)$:
Here, \( p(t) = f(y(t)) \). As we can see above, applying this nonlinear function to a \( T \)-periodic signal results in a \( T \)-periodic waveform. This is in fact a more general property: a function applied to a \( T \)-periodic sinusoid will produce a (more complicated) \( T \)-periodic waveform.

Therefore, we might consider that an arbitrary \( T \)-periodic waveform is related to an underlying \( T \)-periodic sinusoid.

### 2 Fourier Series

The Fourier Series says we can represent any \( T \)-periodic waveform as the sum of many sine waves with frequencies at integer multiples of the fundamental frequency \( f_0 \):

\[
f = 0, f_0, 2f_0, 3f_0, \ldots
\]

where \( f = 0 \) is the DC component, \( f = f_0 \) is the fundamental frequency, and \( f = 2f_0, 3f_0, \ldots \) are the harmonics.

Mathematically, we can write any \( T \)-periodic waveform \( x(t) \) as

\[
x(t) = \sum_{i=0}^{\infty} B_i \cos(2\pi f_0 t + \theta_i) = \sum_{l=-\infty}^{\infty} A_l e^{j2\pi f_0 l t},
\]

where \( B_i \) is a real-valued amplitude, \( \theta_i \) is a real-valued phase offset, and \( A_l \) is a complex-valued coefficient. This is called the Fourier Series representation of the signal. We calculate the coefficients \( A_l \) as

\[
A_l = \frac{1}{T} \int_{0}^{T} e^{-j2\pi f_0 l t} x(t) dt
\]

for each integer \( l \in [-\infty, \infty] \).

We can see that the Fourier Series represents a waveform with an infinite number of sinusoids up to infinitely high frequencies. But what if we only want to represent a waveform with a finite number of sinusoids?

Let us truncate the Fourier Series representation of \( x(t) \) to a summation of \( N = 2M + 1 \) sinusoids and use the fact that \( f_0 = \frac{1}{T} \):

\[
x(t) = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi i}{T} t}.
\]

This format should remind you of the Discrete Fourier Transform, where we are representing a discrete waveform as a summation of discrete sinusoids - except, of course, that the Fourier Series is a representation of continuous waveforms, rather than discrete waveforms. We will explore this connection further below.
3 Questions

1. Periodic Waveforms

Are the following functions $x(t)$ periodic? If so, what is the function’s period $T$ and fundamental frequency $f_0$?

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>Periodic?</th>
<th>$T$</th>
<th>$f_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^t$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e^{jt}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2^{at}$ where $a = a_r + ja_i$</td>
<td></td>
<td></td>
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<tr>
<td>$\frac{dy(t)}{dt}$ where $y(t)$ is $T$-periodic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z(t) \triangleq y(Tt)$ where $y(t)$ is $T$-periodic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C$ where $C$ is a constant</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Answer:

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>Periodic?</th>
<th>$T$</th>
<th>$f_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^t$</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e^{jt}$</td>
<td>Yes</td>
<td>$2\pi$</td>
<td>$\frac{1}{T}$</td>
</tr>
<tr>
<td>$2^{at}$ where $a = a_r + ja_i$ and $a_r \neq 0$</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{dy(t)}{dt}$ where $y(t)$ is $T$-periodic</td>
<td>Yes</td>
<td>$T$</td>
<td>$\frac{1}{T}$</td>
</tr>
<tr>
<td>$z(t) \triangleq y(Tt)$ where $y(t)$ is $T$-periodic</td>
<td>Yes</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$C$ where $C$ is a constant</td>
<td>Yes</td>
<td>any</td>
<td>any</td>
</tr>
</tbody>
</table>

Though $x(t) = C$ can be defined to have any period $T$ and therefore any fundamental frequency $f_0$, when we represent it using sinusoids (as in the Fourier Series) it must be represented with a sinusoid of frequency 0. Therefore constants are usually said to have $f = 0$ and therefore $T = \infty$.

2. Fourier Series and the DFT

In this problem, you will discover how we can use the DFT of $N$ samples of a continuous waveform $x(t)$ to calculate the truncated Fourier Series representation of $x(t)$.

Let’s explore the connection between the truncated Fourier Series and the DFT further. As a reminder, the truncated Fourier Series has us represent continuous, $T$-periodic signal $x(t)$ as

$$x(t) = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{T}it}.$$  \hspace{1cm} (4)

(a) We take $N = 2M + 1$ samples of the $T$-periodic function $x(t)$ across a single period $T$. That is, our samples are at

$$t = \frac{T}{N}k, \ k \in \{0, 1, 2, \ldots, N-1\}.$$
How can we represent the $k^{th}$ sample of $x(t)$, $x_k$?

**Answer:** This gives the discrete samples of $x(t)$

\[ x_k = x\left(\frac{T}{N}k\right) = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{N}ik} = \sum_{i=-M}^{M} X_i e^{j\frac{2\pi}{N}i} \]

(b) Write the relationship between the $x_k$ (samples) and $X_i$ (Fourier Series coefficients) in matrix-vector form. That is, given

\[
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1}
\end{bmatrix} =
A
\begin{bmatrix}
X_{-M} \\
X_{-M+1} \\
\vdots \\
X_{M-1} \\
X_M
\end{bmatrix},
\]

what is the matrix $A$? What is this matrix’s relationship to the DFT matrix, $F_N$?

**Answer:** Rewriting the equation for $x_k$ in terms of $X_i$ gives:

\[
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1}
\end{bmatrix} =
\begin{bmatrix}
1 & e^{j\frac{2\pi}{N}(-M)} & \cdots & e^{j\frac{2\pi}{N}(M-1)} & e^{j\frac{2\pi}{N}(M)} \\
e^{j\frac{2\pi}{N}(2)(-M)} & e^{j\frac{2\pi}{N}(2)(-M+1)} & \cdots & e^{j\frac{2\pi}{N}(2)(M-1)} & e^{j\frac{2\pi}{N}(2)(M)} \\
e^{j\frac{2\pi}{N}(2)(M-1)} & e^{j\frac{2\pi}{N}(2)(M-1)} & \cdots & e^{j\frac{2\pi}{N}(2)(1)} & e^{j\frac{2\pi}{N}(2)(2)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
e^{j\frac{2\pi}{N}(N-1)(-M)} & e^{j\frac{2\pi}{N}(N-1)(-M+1)} & \cdots & e^{j\frac{2\pi}{N}(N-1)(M-1)} & e^{j\frac{2\pi}{N}(N-1)(M)}
\end{bmatrix}
\begin{bmatrix}
X_{-M} \\
X_{-M+1} \\
\vdots \\
X_{M-1} \\
X_M
\end{bmatrix},
\]

This looks similar to the DFT matrix $F_N$, except that it has been reordered and conjugated.

(c) If we reorder the $X_i$ Fourier Series coefficients to be in "DFT order" (that is, in frequency order $f = 0, f_0, 2f_0, \cdots, Mf_0, -Mf_0, \cdots, -2f_0, -f_0$) show that

\[
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1}
\end{bmatrix} = F_N^* 
\begin{bmatrix}
X_0 \\
X_1 \\
X_M \\
\vdots \\
X_{-1}
\end{bmatrix}.
\]

**Answer:** Rearranging the $X_i$ coefficients, we see

\[
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
\vdots \\
x_{N-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & e^{j\frac{2\pi}{N}} & \cdots & e^{j\frac{2\pi}{N}(M)} & e^{j\frac{2\pi}{N}(-M)} & \cdots & e^{j\frac{2\pi}{N}(-1)} \\
1 & e^{j\frac{2\pi}{N}(2)} & \cdots & e^{j\frac{2\pi}{N}(2)(M)} & e^{j\frac{2\pi}{N}(2)(-M)} & \cdots & e^{j\frac{2\pi}{N}(2)(-2)} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
1 & e^{j\frac{2\pi}{N}(N-1)} & \cdots & e^{j\frac{2\pi}{N}(N-1)(M)} & e^{j\frac{2\pi}{N}(N-1)(-M)} & \cdots & e^{j\frac{2\pi}{N}(N-1)(-1)}
\end{bmatrix}
\begin{bmatrix}
X_0 \\
X_1 \\
X_M \\
\vdots \\
X_{-1}
\end{bmatrix}.
\]
We recognize this as the conjugate DFT matrix $F_N^*$ using the conjugacy property that $X_{N-i} = \overline{X_i}$ where $i \in [1,N-1]$. We know that $F_N^* = N F_N^{-1}$, where $F_N^{-1}$ represents the inverse DFT (sometimes denoted as IDFT). In fact, we can write the relation between our $N = 2M + 1$ samples of the continuous waveform $x(t)$ and its truncated Fourier Series coefficients as

$$
\begin{bmatrix}
  x_0 \\
  x_1 \\
  x_2 \\
  \vdots \\
  x_{N-1}
\end{bmatrix} = F_N^* 
\begin{bmatrix}
  X_0 \\
  X_1 \\
  \vdots \\
  X_M \\
  X_{-M} \\
  \vdots \\
  X_{-1}
\end{bmatrix} = NF_N^{-1} 
\begin{bmatrix}
  X_0 \\
  X_1 \\
  \vdots \\
  X_M \\
  X_{-M} \\
  \vdots \\
  X_{-1}
\end{bmatrix}
$$

(9)

or, equivalently,

$$
\begin{bmatrix}
  X_0 \\
  X_1 \\
  \vdots \\
  X_M \\
  X_{-M} \\
  \vdots \\
  X_{-1}
\end{bmatrix} = \frac{1}{N} F_N 
\begin{bmatrix}
  x_0 \\
  x_1 \\
  \vdots \\
  x_M \\
  x_{-M} \\
  \vdots \\
  x_{N-1}
\end{bmatrix}
$$

(10)

3. Fourier Series

Suppose we have the $T$-periodic waveform $x(t)$:

That is, on the interval $[0,T)$, $x(t)$ is given by

$$
 x(t) = \begin{cases} 
  2, & 0 \leq t < \frac{3}{4}T \\
  -2, & \frac{3}{4}T \leq t < T 
\end{cases}
$$

and is $T$-periodic outside $t = [0,T)$. 

(a) Calculate the Fourier Series coefficients $A_l$, given by

$$A_l = \frac{1}{T} \int_0^T e^{-j\frac{2\pi}{T}lt} x(t) dt$$

for integer $l \in [-\infty, \infty]$.

Answer:

$$A_l = \frac{1}{T} \int_0^T e^{-j\frac{2\pi}{T}lt} x(t) dt$$

(11)

$$= \frac{1}{T} \int_0^{\frac{T}{2}} e^{-j\frac{2\pi}{T}lt} x(t) dt + \frac{1}{T} \int_{\frac{T}{2}}^T e^{-j\frac{2\pi}{T}lt} x(t) dt$$

(12)

$$= \frac{2}{T} \int_0^{\frac{T}{2}} e^{-j\frac{2\pi}{T}lt} dt - \frac{2}{T} \int_{\frac{T}{2}}^T e^{-j\frac{2\pi}{T}lt} dt$$

(13)

$$= \frac{2}{T-j2\pi l} e^{-j\frac{2\pi}{T}lt} \bigg|_0^{\frac{T}{2}} - \frac{2}{T-j2\pi l} e^{-j\frac{2\pi}{T}lt} \bigg|_{\frac{T}{2}}^T$$

(14)

$$= -\frac{1}{j\pi l} \left( e^{-j\frac{2\pi}{T}l} - 1 \right) + \frac{1}{j\pi l} \left( e^{-j2\pi l} - e^{-j\frac{2\pi}{T}l} \right)$$

(15)

$$= -\frac{1}{j\pi l} \left( e^{-j\frac{2\pi}{T}l} - 1 \right) + \frac{1}{j\pi l} \left( 1 - e^{-j\frac{2\pi}{T}l} \right)$$

(16)

$$= \frac{2}{j\pi l} \left( 1 - e^{-j\frac{2\pi}{T}l} \right)$$

(17)

Contributors:

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