1. Fundamental Theorem of Solutions to Differential Equations

In this question, you will discover the power of the fundamental theorem of solutions to differential equations. For convenience, we shall restate the theorem here.

**Theorem.** Consider a differential equation of the form,

\[
\frac{d^n y}{dt^n}(t) + \alpha_{n-1} \frac{d^{n-1} y}{dt^{n-1}}(t) + \ldots + \alpha_1 \frac{dy}{dt}(t) + \alpha_0 y(t) = 0
\]

Given \( n \) initial conditions of the form,

\[
y(t_0) = a_0, \quad \frac{dy}{dt}(t_0) = a_1, \ldots, \frac{d^{n-1} y}{dt^{n-1}}(t_0) = a_{n-1},
\]

there exists a unique solution (say, \( f \)).

(a) Consider the following 2 functions.

\[
\phi_1(x) = e^x, \quad \phi_2(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

Prove that \( \phi_1(x) = \phi_2(x) \) by showing that both functions satisfy the following differential equation:

\[
\frac{df}{dx}(x) = f(x) \text{ with } f(0) = 1
\]

**Side note:** Assume that \( 0^0 = 1 \).

**Solution:**

Let’s first look at \( \phi_1(x) \)

\[
\phi_1(x = 0) = e^0 = 1
\]

\[
\frac{d\phi_1}{dx}(x) = e^x
\]

So \( \phi_1 \) satisfies the conditions.

Now let’s look at \( \phi_2(x) \)

\[
\phi_2(x = 0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1
\]
Removing the first term from the series, we get

\[ \phi_2(x = 0) = \frac{0^0}{1} + \sum_{n=1}^{\infty} \frac{0^n}{n!} \]

Zero to any positive finite power is zero, so we can say

\[ \sum_{n=1}^{\infty} \frac{0^n}{n!} = 0 \]

\[ \phi_2(x = 0) = 0^0 = 1 \]

To find the derivative of \( \phi_2(x) \), we can say

\[
\frac{d}{dx} \sum_{n=0}^{\infty} f(x) = \sum_{n=0}^{\infty} \frac{df}{dx}(x)
\]

\[
\frac{df}{dx}(x) = \frac{d}{dx} \left( \frac{x^n}{n!} \right) = \frac{nx^{n-1}}{n!}
\]

\[
\frac{d\phi_2}{dx}(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!}
\]

If we remove the first term of the series, we get

\[
\frac{d\phi_2}{dx}(x) = \frac{0x^{-1}}{1} + \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}
\]

Using the fact that

\[
\frac{n}{n!} = \frac{1}{(n-1)!}
\]

We can say

\[
\frac{d\phi_2}{dx}(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}
\]

If we use substitution and let \( k = (n - 1) \), we get

\[
\frac{d\phi_2}{dx}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \phi_2(x)
\]

Both \( \phi_1 \) and \( \phi_2 \) meet the conditions, therefore they must be the same function

(b) Consider the following 2 functions.

\[ \phi_1(x) = -\sin(x), \phi_2(x) = \sin(-x) \]
Prove that $\phi_1(x) = \phi_2(x)$ by showing that both functions satisfy the following differential equation:

$$\frac{d^2 f}{dx^2}(x) = -f(x) \text{ with } f(0) = 0, \frac{df}{dx}(0) = -1$$

**Solution:**

Initial conditions:

$$-\sin(0) = \sin(-0) = 0$$

So both $\phi_1$ and $\phi_2$ satisfy $f(0) = 0$

Taking the first derivative of each:

$$\frac{d\phi_1}{dx} = -\cos(x)$$

$$\frac{d\phi_2}{dx} = -\cos(-x)$$

Plugging in 0 for $x$:

$$-\cos(0) = -1$$

$$-\cos(-0) = -1$$

Both functions satisfy $\frac{df}{dx}(0) = -1$

Taking the second derivative of each:

$$\frac{d^2\phi_1}{dx^2}(x) = \sin(x) = -\phi_1(x)$$

$$\frac{d^2\phi_2}{dx^2}(x) = -\sin(-x) = -\phi_2(x)$$

Both functions satisfy all three conditions, so $\phi_1(x) = \phi_2(x)$

---

2. **RC Circuit**

Consider the circuit below, assume that when $t \leq 0$, the capacitor has no charge stored ($V_c(t = 0) = 0$). At $t = 0$, the switch closes. Assume that $V_s = 5 \text{V}$, $R = 100 \Omega$, and $C = 10 \mu\text{F}$.

![RC Circuit with Voltage Source](image)

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EECS 16B, Spring 2018, Homework 1
(a) What are the boundary conditions for $I_c(t)$ (i.e. what are $I_c(t = 0)$ and $I_c(t \to \infty)$?)

Solution:

No charge is on the capacitor before time $t = 0$. Using $q = VC$, we know that $V_c = 0$ V before $t = 0$.

At $t = 0$, the switch closes, so we can use KVL to say
\[ V_c + V_R - V_s = 0 \]
Since voltage across a capacitor cannot change instantaneously,
\[ V_c(t = 0) = 0 \]
which means
\[ V_R(t = 0) = V_s \]
From here we can use Ohm’s Law to get
\[ I_c(t = 0) = \frac{V_s}{R} = \frac{5}{100} = 0.05 \text{ A} \]
As $t$ goes to infinity, the capacitor will become fully charged and block all current. Therefore
\[ I_c(t \to \infty) = 0 \text{ A} \]

(b) Use KVL and the relationship between charge and current ($q = \int I dt$) to find the first order differential equation in terms of the current through the capacitor, $I_c$. Assume that $\frac{dV_s}{dt} = 0$. (Hint: You will need to take a derivative with respect to time to get the equation.)

Solution:

From KVL we get:
\[ V_1 + V_R - V_s = 0 \]
Using Ohm’s law and $q = VC$:
\[ \frac{q}{C} + I_R R - V_s = 0 \]
Since $R$ and $C$ are in series, we can say:
\[ I_R = I_c \]
Substituting $q$ in terms of $I$, we get:
\[ \frac{\int I_c dt}{C} + I_c R - V_s = 0 \]
To get the differential equation in terms of $I_c$ and $\frac{dI_c}{dt}$, we take the derivative with respect to time:
\[ \frac{I_c}{C} + R \frac{dI_c}{dt} - \frac{dV_s}{dt} = 0 \]
Since $\frac{dV_s}{dt} = 0$, so we end up with:
\[ \frac{I_c}{C} + R \frac{dI_c}{dt} = 0 \]
(c) What is the eigenvalue $\lambda$ of this equation?

**Solution:**

We can rearrange the equation to

$$\frac{dI_c}{dt} = -\frac{I_c}{RC}$$

Since this is a first order differential equation, $\lambda$ is equal to the coefficient of the $I_c$ term.

$$\lambda = -\frac{1}{RC}$$

(d) Using the eigenvalue and boundary conditions found in the previous parts, find an expression for $I_c(t)$ in terms of $V_s$, $R$, and $C$.

**Solution:**

The general solution to the equation

$$\frac{dy}{dt} = \lambda y$$

is

$$y(t) = Ke^{\lambda t}$$

where $K$ is a constant and $\lambda$ is the eigenvalue of the equation. To find $K$, we can plug in our boundary condition at $t = 0$

$$I_c(t = 0) = Ke^{-\frac{0}{RC}} = \frac{V_s}{R}$$

From this we can see

$$K = \frac{V_s}{R}$$

So our overall equation ends up being

$$I_c(t) = \frac{V_s}{R} e^{-\frac{t}{RC}}$$

We can also double check our answer by looking at the steady state boundary condition:

$$I_c(t \to \infty) = \frac{V_s}{R} e^{-\infty} = 0$$

which agrees with our answer from part (a).
(e) On what order of magnitude of time (nanoseconds, milliseconds, 10’s of seconds, etc.) does this circuit settle ($I_c$ is < 5% of its initial value)?

**Solution:**

The time constant $\tau$ of an RC circuit is just $\tau = RC$. For our circuit:

$$\tau = RC = 100 \Omega \cdot 10 \mu F = 0.001 \text{ s}$$

After 3 time constants, the current will be 5% of its initial value

$$3\tau = 0.003 \text{ s}$$

The circuit will settle on the order of milliseconds.

(f) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

**Solution:**

To reduce settling time, reduce $\tau$. We can achieve this by

i. Lowering the value of $R$ or

ii. Lowering the value of $C$.

Notice how the value of $V_s$ does not change the settling time.

(g) Sketch the current vs. time plot of $I_c(t)$. Make sure to label $I_c(t)$ at $t = 0$, $t = \tau$, $t = 2\tau$, and $t = 3\tau$.

**Solution:**

$$I_c(t) = \frac{V_s}{R} e^{-t/\tau}$$

$I_c(t) = 0.05 e^{-1000t}$

$I_c(0) = 0.05 \text{ A}$

$I_c(\tau) = 18.3 \text{ mA}$

$I_c(2\tau) = 6.76 \text{ mA}$

$I_c(3\tau) = 2.48 \text{ mA}$
3. From Transistors to Inverter

The following circuit is an inverter built with one pMOS and one nMOS transistor. We assume that $V_{out}$ is connected to the input of another identical inverter (not shown). We also assume that there is some capacitance connected between $V_{out}$ and GND. In real transistors, this capacitance arises from a combination of capacitances contributed by the transistors of both the inverter in question and any inverters connected to the output. For the moment, we’ll just call this capacitance $C_L$ (for load). The resistances associated with the pMOS and nMOS are $R_{onP}$ and $R_{onN}$, respectively. Let the threshold voltage of the pMOS be $V_{tp}$, while that for the nMOS be $V_{tn}$. We will now model the action of the inverter as an RC circuit with two switches controlled by $V_{in}$ as what we did in lectures.

(a) For starters, what are the on-off conditions of the two switches? Draw the RC circuit modeling this inverter.

![Diagram](image)

**Solution:**

![Diagram with solution](image)
The upper switch is controlled by the voltage difference between the gate (G) and source (S) voltages of the pMOS. In this circuit, $V_G = V_{in}$ and $V_S = V_{DD}$. The on-condition for pMOS is $V_{GS} \leq V_{tp}$, where $V_{tp}$ is a negative number, around $-0.4$. Usually $V_{DD}$ is greater than $|V_{tp}|$. If $V_G = V_{in} = 0$, $V_{GS} = V_G - V_S = -V_{DD} < V_{tp}$, the switch will be on. If $V_G = V_{in} = V_{DD}$, for pMOS, $V_{GS} = V_G - V_S = 0 > V_{tp}$, the switch will be off.

The lower switch is controlled by the voltage difference between the gate (G) and source (S) voltages of the nMOS. In this circuit, $V_G = V_{in}$ and $V_S = GND$. The on-condition for nMOS is $V_{GS} \geq V_{in}$, where $V_{in}$ is a positive number, around 0.4. If $V_G = V_{in} = V_{DD}$, $V_{GS} = V_G - V_S = V_{DD} > V_{in}$, the switch will be on. If $V_G = V_{in} = 0$, for nMOS, $V_{GS} = V_G - V_S = 0 < V_{in}$, the switch will be off.

(b) Assume that $V_{in} = V_{DD}$ for $t < 0$ and $V_{in} = 0$ for $t \geq 0$. In other words, we are assuming that the input to our inverter can switch states infinitely fast (this is not true in real life but gives us a good lower bound on how fast an inverter can switch). How much energy does it take to fully charge $C_L$? (Hint: $E = \int P \, dt$.)

**Solution:**

When $V_{in} = V_{DD}$ (for $t < 0$), nMOS is on while pMOS is off. $V_{out}$ is shorted to $R_{onN}$, and there is no charge on $C_L$. Then, when $V_{in}$ becomes 0 for $t \geq 0$, nMOS is off while pMOS is on. $V_{out}$ is shorted to $R_{onP}$, and we start to charge $C_L$. The energy spent on charging $C_L$ is $Q_{V_{DD}} = C_L (V_{DD})^2$. Notice that the energy stored on this capacitor is $Q_{V_{DD}}$, but for charging $C_L$, part of the energy is dissipated through the resistor. The power supplied by the voltage source is $VI$, while the power dissipated by the resistor is $IR^2$. To compute the total energy taken to charge $C_L$, you need to integrate power over time:

Here, $I = \frac{(V_{DD} - V_{out})}{R_{onP}} = \frac{V_{DD}}{R_{onP}} \times e^{-\frac{V_{out}}{R_{onP}}}$, so the total energy is

$$\int_0^\infty \left( V_{DD} \times \frac{V_{DD}}{R_{onP}} e^{-\frac{V_{out}}{R_{onP}}} \right) dt = \frac{V_{DD}^2}{R_{onP}} \times (-R_{onP}C_L) \left[ e^{-\frac{V_{out}}{R_{onP}}} \right]_0^\infty = \frac{V_{DD}^2}{R_{onP}} \times (-R_{onP}C_L) [0 - 1] = C_L (V_{DD})^2$$

(c) Given the same condition as in part (b), write down the differential equation that describes $V_{out}(t)$ for $t \geq 0$.

**Solution:**

The current goes through the resistor $R_{onP}$ is the same as the current through $C_L$. Therefore, we could write down the differential equation as

$$C_L \frac{dV_{out}}{dt} = \frac{(V_{DD} - V_{out})}{R_{onP}}.$$

(d) What is the solution to this differential equation? Plot $V_{out}(t)$ for $t > 0$.

**Solution:**

We could decompose $V_{out}$ into a steady state function ($V_{ss}$), and a transient function ($V_{transient}$): $V_{out} = V_{ss} + V_{transient}$, such that $\frac{dV_{out}}{dt} = \frac{dV_{ss}}{dt} + \frac{dV_{transient}}{dt}$. The steady state value of $V_{out}$ is $V_{DD}$ (a constant), because in the long run, the capacitor behaves like an open circuit, while the resistor behaves as a short circuit. That is, $\frac{dV_{out}}{dt} = \frac{dV_{transient}}{dt}$.

The initial state of $V_{transient}$ is $-V_{DD}$, because the initial state of $V_{out} = V_{DD} + V_{transient}$ is 0. Now we could replace the variable $V_{out}$ in the above differential equation with $V_{ss} + V_{transient}$ and then keep only the derivative of $V_{transient}$ at the left hand side, then we will get the following differential equation:

$$\frac{dV_{transient}}{dt} = \frac{-(V_{transient})}{C_L R_{onP}}.$$
The above differential equation looks like we apply some linear operator (derivative) to a function of time, such that the derivative of this function is itself with a scalar. We are finding an eigen-function of this linear operator! The solution to the differential equation should be in the form of $Ae^{st}$, where $A$ and $s$ are some constants. Comparing the desired form and the differential equation, we conclude $s = -\frac{1}{C_LR_{on}}$. The initial state ($t = 0$) for $V_{\text{transient}}$ is $Ae^{0} = A = -V_{DD}$, so $V_{\text{transient}} = -V_{DD} \times e^{-\frac{t}{C_LR_{on}}}$.

Finally, let’s put $V_{\text{transient}}$ and $V_{ss}$ together, so we get

$$V_{out}(t) = V_{DD} - V_{DD} \times e^{-\frac{t}{C_LR_{on}}}.$$  

Here, we set $V_{DD}$ as 5 V and $RC = R_{onP}C_L = 0.05\,\text{s}, 0.1\,\text{s}, 0.2\,\text{s}$.

(e) The term *propagation delay* is used to describe the amount of time it takes between when the input reaches $\frac{V_{DD}}{2}$ and when the output reaches $\frac{V_{DD}}{2}$. Calculate the propagation delay for our inverter above (keep in mind that the input to our inverter changes instantly). Is propagation delay a function of $V_{DD}$?

**Solution:**

Find $t_{pd}$ such that $V_{out}(t_{pd}) = \frac{V_{DD}}{2}$. Then we need $e^{-\frac{t_{pd}}{C_LR_{on}}} = \frac{1}{2}$. Take natural log of the both sides: $\frac{-t_{pd}}{C_LR_{on}} \approx -0.6931$, so we know that $t \approx 0.69R_{onP}C_L$.

In this question, propagation delay is not related to the value of $V_{DD}$. In real devices, increasing $V_{DD}$ does, in fact, speed up the inverter because of transistor effects we have not discussed in this class (namely, increasing $V_{DD}$ has a minor effect on the $R_{onP}$ and $R_{onN}$ of the transistors). For future reference: [http://bwrcs.eecs.berkeley.edu/Classes/icdesign/ee141_f01/Notes/chapter5.pdf](http://bwrcs.eecs.berkeley.edu/Classes/icdesign/ee141_f01/Notes/chapter5.pdf).

(f) Now, consider a serial chain of inverters, each driving the one before it. If we assume that $|V_{in}| = \frac{V_{DD}}{2}$, what is the propagation delay for one of these inverters, given parts (d) and (e)? (If you like, ignore the first inverter and assume that it is driven by an input as in part (a).) Here, we let $R_{onP} = R_{onN} = R$.

**Solution:**

Once the input voltage(output of the previous stage) reaches $\frac{V_{DD}}{2}$, the switch of the inverter in the next stage will turn on/off.

Consider the above case, the propagation delay is the same, $t \approx 0.69R_{onP}C_L$, because when $V_{in}$ changes from $V_{DD}$ to $\frac{V_{DD}}{2}$ (and continue to be 0), pMOS is on, while nMOS is off. Then this circuit starts to
charge $C_L$. Hence the propagation delay is the time spent on charging $C_L$ from 0 to $\frac{V_{DD}}{2}$, which is $\approx 0.69R_{on}C_L = 0.69RC_L$.

Consider the other case: the input voltage is changing from 0 to $V_{DD}$, while initially the capacitor is charged to $V_{DD}$, and pMOS is on, nMOS is off. When the input voltage reaches $\frac{V_{DD}}{2}$, pMOS is turned off, and nMOS is turned on. Then the capacitor starts to discharge.

The differential equation describing $V_{out}$ is

$$C_L \frac{dV_{out}}{dt} = -\frac{V_{out}}{R_{onN}},$$

where the initial state of $V_{out}$ is $V_{DD}$. Following the same procedure in part (d), we could derive

$$V_{out}(t) = V_{DD} \times e^{\frac{-t}{RC_L}}.$$

Now, let’s compute the time needed to reach $V_{out} = \frac{V_{DD}}{2}$; we need $e^{\frac{-t}{RC_L}} = \frac{1}{2}$. Therefore $t \approx 0.69R_{onN}C_L = 0.69RC_L$.

(g) Now let’s consider the following scenario: there are $N$ inverters on the chip in your cell phone. It takes $E$ Joules of energy to charge all of the inverters at once (from zero to $V_{DD}$). What is the value of $C_L$?

**Solution:**
The energy required to charge one inverter is $C_L(V_{DD})^2$, so to charge $N$ inverters together, we need $E = C_L(V_{DD})^2 \times N$. That is, $C_L = \frac{E}{N(V_{DD})^2}$.

(h) Here, we interpret a voltage $V$ as logic “1” when $V > \frac{V_{DD}}{2}$, and logic “0” when $V < \frac{V_{DD}}{2}$. Let’s assume that the maximum frequency, $f$, at which an inverter can switch back and forth between logic “0” and logic “1” at the output is the inverse of the propagation delay (i.e. we can only switch as fast as one propagation delay). Find an expression that links $f$, $C_L$, and $R$.

**Solution:**
According to the answer to part (f), the propagation delay is about $0.69RC_L$, regardless of whether it is from “0” to “1” or from “1” to “0.” Therefore, $f = \frac{1}{t_{pd}} = \frac{1}{0.69RC_L}$.

4. Two Capacitors

Consider the circuit below, assume that when $t \leq 0$, both capacitors are fully charged ($V_1(t = 0) = V_s$ and $V_2(t = 0) = V_s$). At $t = 0$, the switch is closed. Assume that $V_s = 10V$, $R_1 = R_2 = 10k\Omega$, and $C_1 = C_2 = 50\mu F$.

![Figure 2: Two Capacitor Circuit with Voltage Source](image-url)
(a) First, use Kirchhoff’s laws and the capacitor equation \( I = C \frac{dV}{dt} \) to find the second-order differential equation for this system in terms of \( V_2(t) \), \( R_1 \), \( R_2 \), \( C_1 \), and \( C_2 \).

**Solution:**
From KCL, we get:

\[-I_3(t) = I_1(t) + I_2(t)\]

Using the capacitor current equations and Ohm’s law, this equation becomes

\[\frac{V_1(t)}{R_1} = C_1 \frac{dV_1(t)}{dt} + C_2 \frac{dV_2(t)}{dt}\]

In order to get this into terms of only \( V_2(t) \) we use the following substitution:

\[V_1(t) = V_2(t) + I_2(t)R_2\]

\[V_1(t) = V_2(t) + C_2 \frac{dV_2(t)}{dt}R_2\]

Substituting into our original equation and rearranging yields

\[-\frac{V_2(t) + C_2 \frac{dV_2(t)}{dt}R_2}{R_1} = C_1 \frac{d}{dt} (V_2(t) + C_2 \frac{dV_2(t)}{dt}R_2) + C_2 \frac{dV_2(t)}{dt}\]

\[R_1R_2C_1C_2 \frac{d^2V_2(t)}{dt^2} + R_1C_1 \frac{dV_2(t)}{dt} + R_1C_2 \frac{dV_2(t)}{dt} + R_2C_2 \frac{dV_2(t)}{dt} + V_2(t) = 0\]

\[\frac{d^2V_2(t)}{dt^2} + \left(\frac{C_1 + \frac{R_2}{R_1}C_2 + C_2}{R_2C_1C_2}\right) \frac{dV_2(t)}{dt} + \frac{1}{R_1R_2C_1C_2}V_2(t) = 0\]

(b) Now cast this second-order differential equation into the following form:

\[\frac{d\vec{v}}{dt} = A\vec{v},\]

where

\[\vec{v} = \begin{bmatrix} V_2(t) \\ \frac{dV_2(t)}{dt} \end{bmatrix} .\]

**Solution:**

\[\frac{d\vec{v}}{dt} = \begin{bmatrix} 0 & \frac{1}{R_1R_2C_1C_2} \\ \frac{C_1 + \frac{R_2}{R_1}C_2 + C_2}{R_2C_1C_2} & -\frac{1}{R_1R_2C_1C_2} \end{bmatrix} \vec{v}\]

\[\frac{d\vec{v}}{dt} = \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix} \vec{v}\]

(c) Find the eigenvalues of \( A \). Are they real or complex?

**Solution:**

\[\lambda^2 + 6\lambda + 4 = 0\]

\[\lambda = -3 + \sqrt{5}, -3 - \sqrt{5}\]

\[\lambda = -5.236, -0.764\]

The eigenvalues are real.
(d) Using the initial conditions, what is the solution to the differential equation?

**Solution:**

At the time of switching, \( V_1 = V_2 = V_s \) since the voltage cannot change instantaneously across capacitors. However, after switching, \( V_R = V_s \), so \( I_3 = \frac{V_s}{R} = 1 \text{ mA} \) while \( I_2 = 0 \). Our initial conditions are \( V_2(0) = V_s \) and \( \frac{dV_2(0)}{dt} = 0 \). We know that the solution is of the form \( V_2(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \), so using our initial conditions, we can solve for \( c_1 \) and \( c_2 \):

\[
V_2(0) = 10 = c_1 + c_2
\]

\[
\frac{dV_2(0)}{dt} = 0 = \lambda_1 c_1 + \lambda_2 c_2
\]

\[
c_1 = 10 - \frac{10\lambda_1}{\lambda_1 - \lambda_2} = -1.708
\]

\[
c_2 = \frac{10\lambda_1}{\lambda_1 - \lambda_2} = 11.708
\]

\[
V_2(t) = -1.708 e^{-5.236t} + 11.708 e^{-0.764t}
\]

(e) Sketch the voltage vs. time plots of \( V_1(t) \) and \( V_2(t) \).

**Solution:**

\[
C_2 R_2 \frac{dV_2(t)}{dt} = 10^4 \cdot 50 \cdot 10^{-6} \left( -1.708 \cdot \left( -5.236 e^{-5.236t} \right) + 11.708 \cdot \left( -0.764 e^{-0.764t} \right) \right)
\]

\[
V_1(t) = -1.708 e^{-5.236t} + 11.708 e^{-0.764t} + 4.472 e^{-5.236t} - 4.472 e^{-0.764t}
\]

\[
V_1(t) = 2.764 e^{-5.236t} + 7.236 e^{-0.764t}
\]
5. Digital-Analog Converter

A digital-analog converter (DAC) is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the $R$-$2R$ ladder. Here is the circuit for a 3-bit resistive DAC.

Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{DD}$, $V_1 = b_1 V_{DD}$, $V_2 = b_2 V_{DD}$, where $V_{DD}$ is the supply voltage.

As you may have noticed, $(b_2, b_1, b_0)$ represents a 3-bit binary (unsigned) number where each of $b_i$ is a binary bit. We will now analyze how this converter functions.

(a) If $b_2, b_1, b_0 = 1, 0, 0$, what is $V_{out}$? Express your answer in terms of $V_{DD}$.

**Solution:**

There are several ways to solve this problem. For this solution set, we are going to solve for the generic solution rather than solve for each specific case of (a), (b), (c), and (d).

Applying KCL at nodes $V_1$, $V_2$, and $V_{out}$, we get

\[
\frac{V_1}{2R} + \frac{V_1 - b_0 V_{DD}}{2R} + \frac{V_1 - V_2}{R} = 0
\]

\[
\frac{V_2 - b_1 V_{DD}}{2R} + \frac{V_2 - V_1}{R} + \frac{V_1 - V_{out}}{R} = 0
\]

\[
\frac{V_{out} - b_2 V_{DD}}{2R} + \frac{V_{out} - V_2}{R} = 0
\]
Solving this system of equations leads to
\[ \frac{b_2 V_{DD}}{2} + \frac{b_1 V_{DD}}{4} + \frac{b_0 V_{DD}}{8} = V_{out} \]

Plugging in 1, 0, 0 gives the answer.
\[ V_{out} = \frac{V_{DD}}{2} \]

(b) If \( b_2, b_1, b_0 = 0, 1, 0 \), what is \( V_{out} \) ? Express your answer in terms of \( V_{DD} \).

**Solution:**
Plugging into the equation from part (a), we get
\[ V_{out} = \frac{V_{DD}}{4}. \]

(c) If \( b_2, b_1, b_0 = 0, 0, 1 \), what is \( V_{out} \)? Express your answer in terms of \( V_{DD} \).

**Solution:**
Plugging into the equation from part (a), we get
\[ V_{out} = \frac{V_{DD}}{8}. \]

(d) If \( b_2, b_1, b_0 = 1, 1, 1 \), what is \( V_{out} \)? Express your answer in terms of \( V_{DD} \).

**Solution:**
Plugging into the equation from part (a), we get
\[ V_{out} = \frac{7V_{DD}}{8}. \]

(e) Finally, solve for \( V_{out} \) in terms of \( V_{DD} \) and the binary bits \( b_2, b_1, b_0 \).

**Solution:**
From part (a),
\[ \frac{b_2 V_{DD}}{2} + \frac{b_1 V_{DD}}{4} + \frac{b_0 V_{DD}}{8} = V_{out}. \]

(f) Explain how your results above show that the resistive DAC converts the 3-bit binary number \( (b_2, b_1, b_0) \) to the output analog voltage \( V_{DD} \).

**Solution:**
Every increment of \( \frac{1}{8} V_{DD} \) on \( V_{DD} \) represents an increment of 1 to the 3-bit binary number \( (b_2 b_1 b_0) \). For example, if \( V_{out} = \frac{5}{8} V_{DD} \), the input was 5 in binary (1 0 1) → \( (b_2 = 1 \ b_1 = 0 \ b_0 = 1) \).

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